Resolution of Russell's Paradox (reflecting on Logicomix)

Thomas Colignatus
http://www.dataweb.nl/~cool
February 14 2010

Summary

Russell's Paradox is resolved without using a theory of types, allowing a set of all sets. Reflecting on Logicomix: it is a wonderful book but would benefit from a note that Russell's Paradox is resolved, so that readers see better that the story is only a historical episode.

Introduction

Bertrand Russell's paradoxical set is $R = \{x \mid x \notin x\}$. Subsequently $R \in R \Leftrightarrow R \notin R$ and supposedly ("naïve") set theory collapses. Define hoewever $S = \{x \mid x \in S \ \& \ x \notin x\}$ with the small consistency enhancer inspired by Bochenski's discussion [1] of Paul of Venice (1368-1428) [2]. Without contradition we find $S \notin S$. There is no reason for a crisis in the foundations of logic and mathematics and there is no need for a "theory of types" but merely a three-valued logic to determine that R is nonsense (though it has meaning). Observe: (1) a theory of types forbids the set of all sets while it is a useful concept, (2) a theory of types has R in the category "may not be formed" and thus already implies a "third category" next to truth and falsehood. It would be illogical to reject that third category. It is logical instead to generalize that third category to the general notion of "nonsense".

Subsequently, the paradoxes of three-valued logic and the problems of the Gödeliar can be resolved. This analysis is developed in my book "A logic of exceptions" (ALOE). [3] Note that these are original contributions. Also above construction of S is an original construction and I have not seen it elsewhere in the literature. Clearly, ALOE does not provide an axiomatic development, leaving this to others with more time.

A discussion [4] of ALOE is graciously availably by Richard Gill, [5] professor of mathematical statistics at Leiden and member of the Royal Dutch Academy of Sciences, for the Dutch mathematics journal "Nieuw Archief voor Wiskunde". Gill qualifies me as being "eccentric" but that is erroneous and must stem from his English romantic background. Dutch logicians are currently discussing why, how, where – whatever – I would be eccentric and they forget to focus on what logic to teach their students. I would prefer a discussion of logic and some help to find an international publisher for ALOE, currently self-published.

Subsequently there is Logicomix. [6]

---

[1] http://www.questia.com/library/book/a-history-of-formal-logic-by-i-m-bochenski-ivo-thomas.jsp

[2] http://plato.stanford.edu/entries/paul-venice/

[3] http://www.dataweb.nl/~cool/Papers/ALOE/Index.html

[4] http://www.math.leidenuniv.nl/~naw/serie5/deel09/sep2008/reviewssep08.pdf

[5] http://www.math.leidenuniv.nl/~gill/

[6] http://www.logicomix.com/en/

An appreciation of Logicomix

Logicomix is a wonderful read. Anyone who has studied the history of logic will savour the story and interplay of characters. The authors Apostolos Doxiadis and Christos H. Papadimitriou (and artists Alecos Papadatos and Annie Di Donna) acknowledge that they bend history to arrive at a more dramatic storyline but this does not hurt. I didn't see enough of L.E.J. Brouwer but there are other places for when that need arises.

It is not clear for how long the website of Logicomix will be on the internet. The book is a success, likely to become a classic, and fans can imagine a Hollywood production. However, for the sake of this exposition I have included two short sections of the website, in Appendix A and B, retrieved today.

The authors of Logicomix have not benefitted from ALOE which is no surprise given the curious way how logicians do their research. By consequence Logicomix still is very "campy" or old-fashioned and non-ALOE. It displays the chaos and disarray of the field of logic and the foundations of mathematics in that period and even at modern universities, and it does not present that resolution of these matters as given by ALOE.

We should hope that the general public is properly informed and not misled as to the proper situation with respect to Russell's Paradox. For highschool, a mathematical educator [7] would wish for a sound development and training in reasoning, logic and set theory, and this programme seems to be blocked systematically by curious notions of paradoxes, contradictions and complexity. As shown above the issue is straightforward and quite beneficial for young minds eager for some sense and rationality and in danger of nonsense and irrationality.

NB. This present paper focusses on the simple issue of Russell's Paradox since the discussion of the Gödeliar is more complex. It follows Logicomix in selecting Russell and his paradox as keys to open the mind. I thank the authors for this tactic.

Conclusion

Logicomix recalls Russell's note to Frege and Frege's hurried note for the printer of his proud but suddenly "destroyed" book. The authors of Logicomix might regard ALOE in the same way. Logicomix seems to be conceived to tell the everlasting story of logic and the foundations of mathematics but turns out to be only a report on a historical episode. Readers who enjoy Logicomix as much as I have done may hope that the authors also have the insight in logic and the intellectual strength as shown by Frege.

Appendix A: "The story"

Quote from the website: [8]

"Covering a span of sixty years, the graphic novel Logicomix was inspired by the epic story of the quest for the Foundations of Mathematics.

This was a heroic intellectual adventure most of whose protagonists paid the price of knowledge with extreme personal suffering and even insanity. The book tells its tale in an engaging way, at the same time complex and accessible. It grounds the philosophical struggles on the undercurrent of personal emotional turmoil, as well as the momentous historical events and ideological battles which gave rise to them.

The role of narrator is given to the most eloquent and spirited of the story's protagonists, the great logician, philosopher and pacifist Bertrand Russell. It is through his eyes that the

---

[7] http://www.dataweb.nl/~cool/Papers/Math/Index.html
[8] http://www.logicomix.com/en/index.php?option=com_content&view=article&id=92&Itemid=28

plights of such great thinkers as Frege, Hilbert, Poincaré, Wittgenstein and Gödel come to life, and through his own passionate involvement in the quest that the various narrative strands come together."

Appendix B: "The problem"

The following quote is from the website, section "Topics", and gives the footnote on what Russell's Paradox is about. [9] Note the distinction between "contradiction" and "paradox". In two-valued logic $R \in R \Leftrightarrow R \notin R$ is a contradiction. A paradox is a "seeming contradiction", i.e. something that in one respect seems like a contradiction but in another respect is consistent (though not necessarily true). For two-valued logic, Russell's Paradox is a contradiction. In three-valued logic, Russell's set is nonsensical and the statement $R \in R$ has truthvalue "nonsense" or ½. Given the consistency of that observation, Russell's Paradox is aptly called a "paradox". However, some authors use the word "paradox" sometimes to *only* mean a contradiction and this might be the case in Logicomix.

"(Notebook, Logicomix) Discovered in 1901, as Russell was working on his first book on the foundations of mathematics, the Principles of Mathematics (published in 1903), the Paradox, in the form originally expressed, shows an essential flaw in Cantor's set theory, developed from Bolzano's simple concept of a "collection of elements with a common property". By the generality of this definition, which Frege extended to the realm of logic, one can speak of a "sets of sets" and thus, eventually of the "set of all sets". Of the elements of this all-encompassing set one defines the property of "self-inclusiveness", i.e. of a set containing itself as an element. Thus, for example, the set of all sets is a set (and thus contained in itself), as is the set of all entries in a list (it can appear as an entry in a list), but the set of all numbers is not a number and thus not contained in itself. By virtue of this property, we can define the "set of all sets which don't contain themselves", and ask, with the young Russell, the question: "Does this set contain itself or not?" See what happens: if it does contain itself, it follows that it is one of the sets which don't contain themselves (as this is the property that characterizes elements of this set) and thus cannot contain itself. But if it doesn't contain itself, then it does not have the property of not containing itself, and thus does contain itself. This situation, in which assuming something implies its negation, and vice versa, is called a paradox. When a paradox, such as Russell's, arises in a theory, it is a sign that one of its basic premises, definitions or axioms is faulty."

---

[9] http://www.logicomix.com/en/index.php?option=com_content&view=article&id=92&Itemid=28