Risk in the LogNormal distribution

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The following is a new section in *The Economics Pack. Applications of Mathematica.*

XX.X Risk in the LogNormal distribution

xx.x.1 Summary

The `ProductRisk` package provides some routines and insights for the notion of *risk* in the LogNormal distribution. This notion of risk arises when outcomes are interpreted as growth factors.

> **Economics [ProductRisk]**

xx.x.2 Introduction

xx.x.2.1 The subject

The subject of discussion are *rates of growth* or *rates of return*. For readers familiar with this subject - who may have read Luenberger (1998) "Investment Science" - the following points can be directly mentioned. When you are not familiar with this subject then be patient with the terms that you don't recognize at first since they should become clear in the remainder.

On the one hand there is the *lognormal distribution* with two parameters on mean and spread and on the other hand there is a binary *prospect* with three parameters on winning, losing and the probability of a win. The question is how these relate to each other. Comparing two with three parameters seems like begging for problems, especially when those three may be linked by a hidden condition. The question however arises naturally in investment theory. There appears to exist an important distinction:

- ■ The lognormal and prospect can hold for an undefined length of the unit period. They then represent a summary of the long term state. When we keep the mean and the probability of winning the same then the spread in the prospect will not be the same as the spread in the lognormal. This is the problem of having two versus three parameters. Risk, defined as probable loss, however will be the same in both. Thus the prospect is a good summary of the lognormal in terms of mean, odds and risk. Here we will use the Prospect[...] object.
- ■ The prospect can also be seen as a wheel of fortune. In repeated games, the prospect gives the parameters for each single step but the long run results will be given by a lognormal distribution. Conceivably the time unit of the step is undefined as well but in diffusion the parameters of the lognormal are dependent upon time. Here we will the EProspect[...] object.

This distinction is clear in itself but the important insight is that the translation really depends upon it. The translation for the first is different than for the second. For the first area we know that two versus three parameters is begging for problems so that the spread may easily lose out. For the second area it may be that the spread is the important parameter. The distinction between Prospect and EProspect then is important.

The Prospect[Profit, -Loss, p] object holds for the single period and additive world and the (exponential) EProspect[u, d, p] for the repeated case and multiplicative world. Here Profit, Loss $\geq$ 0 and growth factors u (up) $\geq$ 1 $\geq$ d (down) > 0. The rule is that the EProspect object can translate to a Prospect with (*factor* - 1) or Log[*factor*]. Growth rates can indeed be expressed ambiguously both as perunage rates or as powers of e (called continuous compounding). While the rates have different values the factors remain the same and thus these factors form a good basis for a general approach. Thus $\mu = E[x]$ and σ will stand for factors as well and the lognormal will be parameterized with θ and τ.

The focus of the present discussion is how to translate the parameters of single period Prospect[u -1, d - 1, p] and the multiplicative EProspect[u, d, p] into the parameters of the lognormal distribution with parameters θ and τ, and vice versa. It may be numerically simple to translate a Prospect into an EProspect object but conceptually that does not mean that the factors in that Prospect of the long run state will also be the factors for an actual wheel of fortune.

xx.x.2.2 Questions that motivated this notebook and package

This notebook and package have been motivated by these questions:

■ Let x = *new* / *old* be a growth factor. Rates of growth are either calculated relatively as r = x - 1 or by using logarithms Log[x]. Obviously x - 1 ≈ Log[x]. When the long run state is summarized with a lognormal distribution so that we should be thinking in logs then it is a bit curious that we still may ask what the expected percentage rate of growth is. It is easier for the human mind to calculate a relative rate than a logarithm so it may be that logarithms are the true model and the relative rates are only a useful tool for communication. But we wonder what the error is and whether more can be said.

■ The distinction between a single period investment and a repeated investment is a bit unclear since the definition of the length of the period is arbitrary. The parameters would also be averages over a longer period - so the single period cannot do without the repeated case. Is the single period only a teaching tool ? For the single period we have the CAPM mean-variance approach that balances the spread to the expected value. For the repeated investment we balance the volatility with the maximal growth portfolio. Are these approaches consistent, in particular when the period is arbitrary ?

■ The growth model assumes that the rates are distributed with a normal distribution so that the price is distributed with a lognormal distribution. The expected value of the price of the stock then has a correction term that depends upon the volatility or the returns. How is this exactly, and why ? How does this relate to the plain old expected value of the single period ? When the unit of period is arbitrary, should we not have $\mu[1]$ = E[x] = μ ?

■ There is a notion of risk as ρ = - E[x < 0] for the single period choice, i.e. the probable loss, and conditional risk κ = - E[x | x < 0] = ρ / (1 - p) = Loss as the expected loss when you know that you will lose, with p the probability of success. How does this translate to the repeated choice situation ? The repeated game has factors of growth that are always x > 0 so they are not risky in that earlier sense. It is kind of strange that investment does not come with risk. Do we need to extend our definition of risk ?

■ Luenberger 1998:417 has the example of a wheel of fortune. This example has been discussed in the notebook on probability in The Economics Pack User Guide and we revisit it now. One section on the wheel has probability 1/6 and it pays out $6 for every $1 bet. In terms of expected value we would not be interested in this since on average we just get back what we gambled. However, the optimal growth strategy uses the section as a "hedge". There is even a section on the wheel that has a worse performance but it can still be a hedge. Why is that ? Is it really "volatility pumping" ? Why is our exposure to risk not relevant ? There is a

range of optimal strategies that have the same rate of growth but when we compare {1/2, 1/3, 1/6} and {5/18, 0, 1/18} then in the first we risk all our capital and in the second only 1/3. Why does this not matter ? Also, when we try some other wheels of fortune then the normal equations break down. PM. There are two varieties of the wheel of fortune. The latter allows us to put bets on the fields separately. The kind that we will discuss here is "play or not" with a fixed allocation. This situation would also hold for an optimal allocation of the first case.

- The binomial lattice on a binary prospect generates a lognormal distribution - but does this hold for all prospects ? An obvious question is whether the lognormal distribution can be projected conversely into a binary prospect. Luenberger (1998:315) provides such a projection but his approach causes some questions. He presents two equations with variables d, u, p plus the two parameters θ and τ from the lognormal distribution. He then chooses $d = 1 / u$ and solves u and p in terms of the lognormal parameters. However, if there is an interpretation of p for the lognormal distribution then it already follows from θ and τ.

The focus of this notebook is on the translation of the prospects and the lognormal. Around this we will discuss these other questions to illuminate what all it means.

Our main question is whether the single period expection, i.e. the good old expected value of the prospect, still has an explicit meaning with respect to the long run expectation of the growth process. The standard approach has the suggestion that there is a difference between $\mu[1]$ and $\mu = E[x]$, with the implication that the theory of expected value is only for textbook teaching since the true world is a repeated game.

xx.x.2.3 The major result

With respect to that main question the standard view appears to have a focus on repeated games with the EProspect object and a neglect of the long run state with the Prospect object. The translation of a lognormal into a Prospect solves that question. The long run state is given for an undefined length of the unit period and hence there actually is no $\mu[1]$. The factors are not immediately relevant for a repeated game.

The result on the long run state also has an implication for repeated games. The standard approach derives from a specific choice of $\{\theta, \tau\}$ using formulas for the rates. It appears that there is a different choice with $\{\mu, p\}$ and implied $\{\theta^*, \tau^*\}$, just like the long run state. The notions of Risk and ProductRisk widen the field of criteria to determine what is best in what circumstance.

When $p = 1/2$ then in the lognormal model $\theta = 0$ and this standardly requires $u = 1 / d$. This EProspect however does not satisfy that restriction but the formula on the rate of course generates an estimate.

q = EProspect[1.05, .97, 0.5];

ToENMT[Log, q]

LogNormalQ ::*neq* :

  EProspect Pr *0.5* *not equal to Pr* *0.591462* *implied by Theta and Tau*
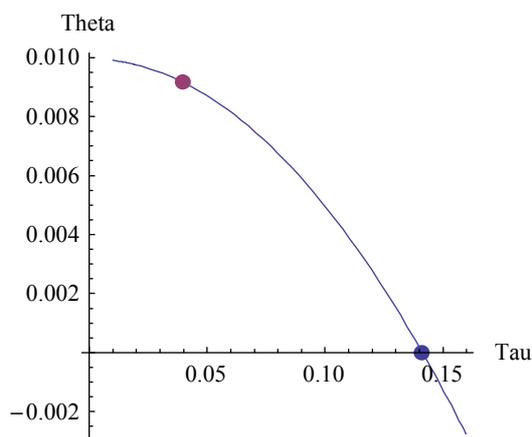
Hold[ExpNormalMuTau][1.01, 0.0396247]

% // ReleaseHold

LogNormalDistribution[0.00916548, 0.0396247]

- The purplish dot gives the $\{\theta, \tau\}$ outcomes from the standard formulas for the rates. The blue dot gives the $\{\theta^*, \tau^*\}$ outcomes from using $\{\mu, p\}$. The dots are on the line $\mu = p\, u + (1 - p)\, d$ (in another dimension).

ToENMT[Mean, Plot, q, {d, .85, 1}]

xx.x.2.4 The approach in the following

To prevent confusion it is useful to start with the parameterization of the lognormal distribution. Our notation is consistent with the rest of The Economics Pack and differs a bit from the one used by Luenberger (1998).

Subsections up to 6 are notations, reformulations, pinning down of essentials, and creation of an environment in *Mathematica*. Subsection 7 and 8 discuss the long run state, basically using Prospect. The EProspect object is used here as a convenient format for the

lognormal distribution but this use should not confuse us into thinking that it represents the parameters for a repeated game. Here we can write μ, θ, τ and p. Subsection 9-12 discuss the repeated game. We write θ and τ as the derived parameters of the EProspect so that the $\{\mu, p\}$ now associate with θ^* and τ^*.

The Economics Pack User Guide already contains a notebook on probability that also creates the Draw routine to simulate a wheel of fortune. The Luenberger (1998) example is recreated there too. The Prospect has already been discussed in the notebook on risk in the User Guide and thus we can concentrate here on the EProspect as the new development. The next step is the use of the binomial lattice that brings us to the lognormal distribution. The wheel of fortune is an intended application to reconsider but the present discussion is already long enough.

The appendix contains a discussion on the choice of the unit period. An EProspect can be fitted into a lognormal distribution by adjusting the length of the period. But this appears to have limited scope and does not fit itself with our notion that the choice of the period is arbitrary - so that it should be possible to use a prospect as a summary of a longer run result.

There is no conclusion section since that has already been given in this introduction.

xx.x.2.5 A note on the routines

The routines fall in three parts. Some general, some for the long run state and some for the repeated game situation. But there is an overlap when some of the approach of the long run can also be applied to the repeated games. The division between Prospect and EProspect is not always practical since in this context it is frequently best to work with factors. The label "Baseline" stands for the long run state. The label "Log" stands for the repeated games that use the θ and τ estimated on the logs. The label "Mean" can be applied to both the long run and the repeated games and relies on using μ and p. The label ENMT stands for the ExpNormalMuTau[μ, τ] version of the lognormal distribution.

- *General:* ENMT[μ, τ], ENMTTheta[μ, p], ENMTTau[μ, p], ERisket[], ProspectEV[q], SinglePeriodEV[q]
- *For the long run stat*e: ToBaselineEProspect[μ, τ], ToERisket[μ, τ, Method → Baseline], ENMTImpliedEstimates[μ, τ], ENMTErrorOnMu[q]
- *For repeated games* using logs: ToENMT[Log, q], ProductRisk[q], LogNormalToEProspect[θ, τ], ToLogEProspect[μ, τ], ENMTImpliedFactors[μ, τ], ToERisket[μ, τ, Method → Log], LogNormalQ[Log, q].

  ■ *Routines from the long run state that are also relevant for repeated games:*
    LogNormalFitQ[Mean, q], LogNormalQ[Mean, q], ToENMT[Mean, q]

xx.x.3 Parameters of the LogNormal distribution

Let S be our stochastic variable such as the price of a stock or our wealth. In general Log[$S[t]$] ~N[Log[$S[0]$] + $\theta\, t$, $\tau\, \sqrt{t}$], with a start value, a parameter θ for the expectation of the logs and τ for their volatility. Now we just compare $t = 0$ with $t = 1$ in discrete fashion - see Luenberger (1998:309-311, equation (11.20)) on discrete simulation. We normalize $S[0] = 1$ so that Log[S] ~N[θ, τ]. Thus we start out at the beginning of the period with capital 1 and at the end of the period the capital is S.

For S we have the mean $\mu = E[S]$ and the spread σ. Let $\theta = E[\text{Log}[S]]$ and τ the spread of Log[S]. Thus $S \sim$ ExpNormal[μ, σ] when Log[S] ~ Normal[θ, τ].

  ■ We will use the expression ExpNormal[μ, σ] = LogNormal[θ, τ] to emphasize that μ belongs to the realm of S and not Log[S]. The convention is to say that S is distributed with LogNormal[θ, τ].
  ■ We find that θ is the log of the median value of S. i.e. $\theta = \text{Log}[\text{Median}[S]]$ or Median[S] = e^{θ}, and that $\lambda = \text{Log}[\mu] = \theta + \tau^2/\,2$ or $\theta = \text{Log}[\mu] - \tau^2/\,2$. It is also useful to see the formula $\mu = \text{Median}\, \text{Exp}[\tau^2/\,2] = \text{Exp}[\,\theta + \tau^2/\,2\,]$.
  ■ PM. Luenberger (1998) uses E[S] and not our $\mu = E[S]$, μ for our λ, ν for our θ, σ for our τ.

  ■ The format of the lognormal distribution that is most applicable for growth factors is a mixture that uses both the mean μ of the factors, with a value around 1, and the volatility τ of the rates of return. This is the format ExpNormalMuTau[μ, τ] and we will use the abbreviation ENMT[μ, τ] for this. See also FromMuTau[μ, τ].

    ExpNormalMuTau [μ, τ]

$$\text{LogNormalDistribution}\left[\log(\mu) - \frac{\tau^2}{2}, \tau\right]$$

A process like this is essentially a bit strange since it allows us to gamble with our wealth, possibly reducing to the tiniest fraction of a cent along the way, without ever facing the possibility of ruin, which is hardly realistic. Study of this case nevertheless is basic in the road to understanding the process.

There is a distinct difference between the mean θ of the logarithms and the logarithm of the mean Log[μ]. In other words there is a difference between the mean rate of returns and

the rate of return on the mean. Since $S > 0$, a change of τ must have an impact upon μ. This is given by $\mu = \text{Exp}[\theta + \tau^2 / 2]$. This also expresses that we need only know the rates (and their implied volatility) to find the level mean. In statistical practice we have a series $\{S[0], ..., S[n]\}$ and then the determination of the μ is not obvious. The procedure is rather that we consider the expected value of the rate of return $\theta = \text{E}[\text{Log}[S]]$ and the volatility τ. From this we calculate $\text{Log}[\mu]$ and $\text{E}[S] = \mu$. Thus μ is really the interesting concept for the end of the period. In the proportional method we would calculate $\mu / 1 - 1 = \mu - 1$ as the expected rate of return. With the logarithmic growth measure this is $\text{Log}[\mu]$. Thus our wealth grows with the mean rather than θ. The θ is only the expectation of the random variable $\text{Log}[S]$ in terms of its normal distribution but it is not the relevant variable in terms of expection of the lognormal distribution. Thus we better write $\theta = \text{E}[\text{Log}[S] \mid \text{Normal}]$ to indicate its limited area of relevance.

- Conclusion 1: We are interested in μ and $\text{Log}[\mu]$ and not in θ. Also in the scenario with the maximal growth rate and the frontier of choice we should use μ rather than θ.
- Conclusion 2: However, we often use θ and τ to find μ.
- Conclusion 3: The correction of θ by τ to find μ is less mysterious now. And the correction should not be forgotten, i.e. by a wrong focus on θ instead of μ.
- Conclusion 4: It is practical to parameterize the distribution in terms of $\{\mu, \tau\}$ as well next to $\{\theta, \tau\}$ (keeping the same τ).

xx.x.4 Binary Prospect

The default prospect format is Prospect[Profit, -Loss, Pr] with Pr the probability of Profit. It is defined with only implicit reference to our start capital. It is applicable to the single period, at least. Let Profit stand for the profit rate and Loss for the loss rate. We can determine the expected value, risk $\rho = -\text{E}[x < 0]$ and spread. Note that Loss is a condidtional risk, via $L = \rho / (1 - p) = -\text{E}[x \mid x < 0]$.

```
q = Prospect[Profit, -Loss, Pr];
```

```
ProspectEV[q]
```

Pr Profit − Loss (1 − Pr)

```
Risk[q, Position → True]
```

Loss (1 − Pr)

Simplify[Spread[q], Assumptions −> {Profit > 0, Loss > 0}]

$$\sqrt{-(\mathrm{Pr}-1)\,\mathrm{Pr}}\ (\mathrm{Loss}+\mathrm{Profit})$$

See The Economics Pack User Guide for a development of these prospects and this notion of risk.

xx.x.5 Binary EProspect

xx.x.5.1 Definition

The Prospect object has been defined in terms of nonnegative variables Profit and Loss so that the minus sign becomes explicit, as Prospect[Profit, -Loss, p]. It is useful to keep this binary distinction for the multiplicative factors. Logically seen, growth factors are nonnegataive, and a prospect without a negative element is not the Prospect object that we used up to now. Thus we create the EProspect for the factors in multiplicative processes.

Let there be a nonnegative rate of growth g so that the *up* growth factor is $u = 1 + g$. Let there be a nonnegative rate of failure f^* such that the *down* factor is $d = 1 / (1 + f^*) = 1 - f$. with $0 < d \le 1$. Thus both $g \ge 0$ and $f^* \ge 0$. The probability of ruin $d = 0$ is a case that is not in focus now. Let the probability of nonnegative growth be p so that the probability of negative growth is $1 - p$. The random variable $R = 1 + r$ takes values u or d with said probabilities. The mean factor is $1 + \bar{r} = e^\lambda$ for some $\lambda = \mathrm{Log}[1 + \bar{r}]$. A process like this can be called a binary multiplicative prospect.

- EProspect[u, d, p] gives the formal binary multiplicative prospect when outcomes are growth factors $R = 1 + r = e^y$, with r the rate of growth and parameters $0 < d \le 1 \le u$. The probability for profit is p. This is the default EProspect. Now there are no negative entries.

  $EProspect[]

  EProspect(ProfitFactor, LossFactor, Pr)

- The following is an example where we are certain to lose. See the discussion below how this expected value has been computed. ProspectEV works on both Prospect and EProspect but treats them differently.

  ProspectEV[EProspect[1, .95, 0]]

  0.95

```
EProspect [profit factor, loss factor, PrProfit]

            is an object that contains the key data with  0 < loss factor ≤ 1 ≤
            profit factor of a multiplicative Prospect were outcomes are x = E^y


$EProspect []                    gives the binary prospect format,
                                 for multiplicative cases with outcomes E^x


$EProspect [                     gives the assumptions
Assumptions ]                    rule that can be used in Simplify
```

xx.x.5.2 A wheel of fortune

The EProspect represents a wheel of fortune. After each run we can normalize our capital again to 1 and face the same EProspect. Multiplication of the outcomes gives the final result. Let us run the wheel of fortune for n times, repeatedly gambling all our winnings again and get a series $u\ d\ d\ d\ u\ ....\ u\ d\ u\ u = u^m d^{n-m}$. The average return factor is $1 + \hat{\bar{r}}$ $= (u^m\ d^{n-m})^{1/n}$ and $\hat{p} = m/n$. This is only an estimate based upon a random outcome. What is the expectation ? It is tempting to think that it is $u^p d^{1-p}$ but this is only the median.

- ▪ There is the binomial lattice for two steps. In this graph there is an alphabetic sorting so that *d* is on top and *u* is in the bottom. Nevertheless it is clear that expected outcomes can be added columnwise.

  BinomialLattice[Plot, EProspect[u, d, p], 2, Do → Times]
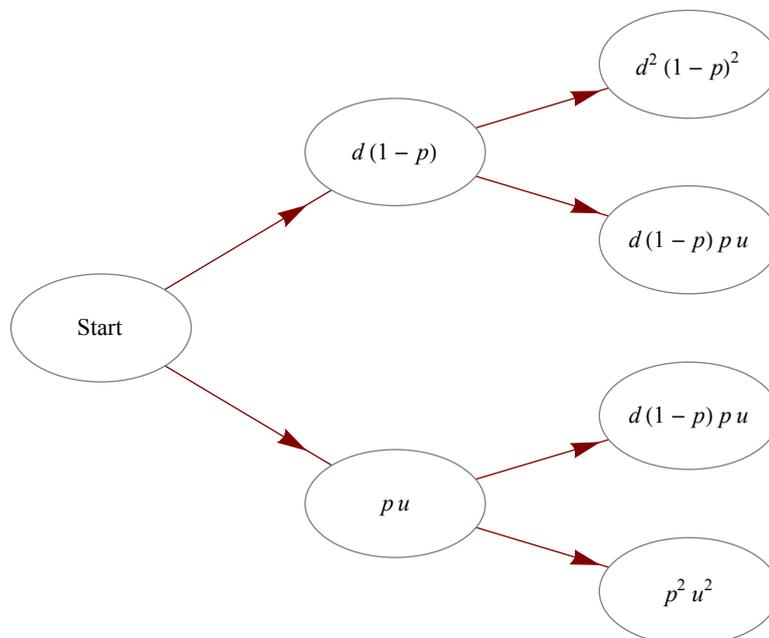

```
BinomialLattice[Plot, e_EProspect, n_Integer]
```

                                        plots the lattice for the EProspect with depth *n*

```
BinomialLattice[
Do, {x, y}, n_Integer]
```
                                        gives the list of arrows
                                        and labels for powers of *x* and *y*

Display options are passed on to TreeForm. Own options are: **Do**: Automatic (default, chooses up and down), Pr (chooses p), Times (multiplies); **Disk**: None, Automatic or fixed numerical size.

xx.x.5.3 The single period and repeated games

For one period ahead clearly $\mu[1] = 1 + \bar{r} = e^{\lambda} = p\,u + (1 - p)\,d$. The question arises whether we can find a lognormal distribution that generates $\mu[\theta, \tau] = 1 + \bar{r}$ as well.

For the multiperiod game the expectation quickly becomes more complex. The outcomes are the possible powers of the factors, weighted by the binomial distribution. Over time the outcome will traverse along the real axis and the binomial distribution can be approximated by a normal distribution. There are two important considerations here. (1) The higher powers of d will move to zero and the higher powers of u will have a high impact so that one can imagine the influence of the spread on the mean. (2) Properly seen, we should weigh the levels with the normal distribution and not the logs. However, we are not interested in the higher powers but in the period average. When we normalize by taking logarithms then there is a direct appeal to the law of large numbers for the rates of return. When we normalize each period to 1 then we actually take $\Delta\text{Log}[S[t]] = \text{Log}[S[t]] - \text{Log}[S[t - 1]]$ and the normal distribution would hold for these rates.

The normal distribution for x is actually also a normal distribution for rates $r = x - 1$. Luenberger (1998:311) mentions on the choice of the normal or lognormal: "The two methods are different, but it can be shown that their differences tend to cancel in the long run. Hence in practice, either method is about as good as the other." This is a strong notion especially when we don't know what the real process is (except under simulation).

The use of logs thus derives from: (1) Elegant results in modeling. (2) The true model may be logs. Our convention to use proportional rates *new / old* - 1 may well be a quick and dirty way to approximate logarithms, primarily since the human mind is not particularly good at transforming to logs. Thus a good rule is that we will consider the true process in terms of logs but translate results occasionally towards proportional rates for the necessary communication of what actually happens.

For the multiperiod setting we find $\theta = \text{E}[\text{Log}[S] \mid \text{Normal}] = p\ \text{Log}[u] + (1 - p)\ \text{Log}[d]$ with $p = m / n$ at the population limit, thus with the *median* $= u^p d^{1-p}$. An expression for τ gives also a value for μ. The subsequent question is how that long run version compares to the one period version.

xx.x.5.4 What we would like to see and what, it seems, cannot be done

For EProspect[*u, d, p*] we have the expected value for a single period $\mu[1] = p\,u + (1 - p)\,d$. We would like to extend this to a process μ^t in which the variables *u, d* and *p* still play a role in the sense that they provide a summary. Since $\mu^t = p\,u^t + (1 - p)\,d^t$ holds for $t = 1$ it cannot hold for all *t*. Repeated games bring us to the binomial lattice and the lognormal distribution. But $\mu[\theta,\tau]$ based on the lattice will, in the current view, differ from $\mu[1]$. Thus the expected value for a single game differs from the long run expectation. Nevertheless it still is a useful idea that the prospect is used to summarize that long run situation. These two notions seem in conflict with each other.

- This equates $\mu[1] = p\,u + (1 - p)\,d$ and $\mu[\theta,\tau] = E[S]$ when the multiplicative prospect is interpreted in lognormal fashion. Let $\theta = E[Log[S]]$ and τ the volatility. The equation clearly imposes a restriction on the parameters.

  $$p\,u + (1 - p)\,d\ ==\ \mathbf{Exp}\!\left[\theta + \tau^2/2\right]$$

  $$d\,(1 - p) + p\,u = e^{\frac{\tau^2}{2} + \theta}$$

- This gives the standard explanation of the lognormal θ and τ. Setting this equal to $\mu[1]$ is rather impossible.

  $$\mathbf{ProspectEV[\,EProspect[u,\ d,\ p]\,]}\ ==\ \mathbf{Exp}\!\left[\theta + \tau^2/2\right]$$

  $$d^{1-p}\,e^{\frac{1}{2}\,(1-p)\,p\,\log^2\!\left(\frac{u}{d}\right)}\,u^p = e^{\frac{\tau^2}{2} + \theta}$$

Apparently we may try to find another θ^* and τ^*, i.e. another translation of EProspect[*u, d, p*] into a lognormal format.

xx.x.5.5 Conclusions

- Conclusion 1: For one period ahead $\mu[1] = 1 + \bar{r} = e^{\lambda} = p\,u + (1 - p)\,d$. This is a fundamental property of the model. Note that the right hand side indeed is an expected value, albeit over a single period. This determination of the expectation should for example not be confused with the estimation procedure when we determine $\hat{\mu}$ from $\hat{\theta}$ and $\hat{\tau}$.

- Conclusion 2: For the multiperiod setting the natural conclusion with the law of large numbers is that $Log[S] \sim N[\theta,\tau]$ and hence *S* is lognormal. The use of logarithms take precedence over relative rates $x - 1$ (but it still may be that the true model is in relative rates).

- Conclusion 3: The issue is a matter of growth accounting. We have not discussed utility but have not needed so. Utility arises only when *balancing* expected value, volatility and risk.

- Conclusion 4: For the multiperiod setting $\theta = p \, \text{Log}[u] + (1 - p) \, \text{Log}[d]$. Hence the *median* $= u^p d^{1-p}$ and this is not E[S]. Depending upon τ there will be another $\mu[\theta, \tau]$ than $\mu[1]$ for the single period.

- Conclusion 5: It remains to be seen whether this θ also holds for the one period setting.

- Conclusion 6: Nevertheless it still is a useful idea that the prospect is used to summarize that long run situation. Thus there is an issue of reconciling $\mu[1]$ and $\mu[\theta, \tau]$. The latter may only fit with $\mu[1] = p \, u + (1 - p)$ for special values of u and d.

xx.x.6 Comparing the Prospect and the EProspect

Consider the translation of the EProspect into the one period Prospect. The Prospect[Profit, -Loss, Pr] is not defined in comparison with the start capital but it can be used such. In finance, we normalize our beginning capital to 1. When we look one period ahead then we actually have a simple (non-E) prospect.

- Merely accounting gives Profit $= u$ - 1 and -Loss $= d$ - 1 and thus Prospect[u - 1, d - 1, p] = Prospect[g, -f, p] with the rates of growth. This is proportional growth accounting that includes division by 1, with $r / 1 - 1 = r - 1$.

- Logarithmic growth accounting gives Prospect[Log[u], Log[d], p]. This is similar to the first since for small numbers Log[x] $\approx x$ - 1 but with different derivatives.

The (non-E) Prospect[u, d, p] and 1 + Prospect[g, f, p] are the same in expected value but in a certain sense different in terms of risk that concerns the danger of negative values.

- The multiperiod prospect has a somewhat hidden risk with respect to our start capital. When you are used to these issues then it is clear but when you start studying them then it is important to emphasize it.

```
q = Prospect[u − 1, d − 1, p];
```

```
evSingle = ProspectEV[q]
```

$(d - 1)(1 - p) + p(u - 1)$

riskSingle = Risk[q, Position → True]

$-(d - 1)(1 - p)$

- Prospect[Log[u], Log[d], *p*] has the same structure. Here the risk is not hidden since Log[*d*] would be negative. However, the notion of Risk for the lognormal distribution has not been defined yet in terms of above concepts. NB. This object is a Prospect and not an EProspect.

qlog = ToUtility[Prospect @@ $EProspect[]] /. Utility → Log /. Pr → p

Prospect(log(ProfitFactor), log(LossFactor), *p*)

medianRepeated = ProspectEV[qlog]

$(1 - p) \log(\text{LossFactor}) + p \log(\text{ProfitFactor})$

riskRepeated = Risk[qlog, Position → True]

$-(1 - p) \log(\text{LossFactor})$

This analysis repeats part of what we already concluded with respect to the median but it extends on the element of risk. Why play the multiperiod game when there is this risk ? There are two possible answers: (1) We have no choice. Life presents us with choices and we cannot opt out. (2) We have a strategy that accepts risk with the hope for good return. NB. The EProspect studied here may itself result from a more basic EProspect *and* the strategy. By keeping always some capital in reserve we gain access to the next round of playing.

- Conclusion 1: There is scope for a developmentor Risk (other than volatility) in the lognormal setting, even though officially all results are nonnegative. Such a notion of risk may help decide between strategies.
- Conclusion 2: The single period Prospect actually helps out for the multiperiod EProspect when we transform the variables, either substracting 1 or taking logs.
- Conclusion 3: For the single period, the expected value of *proportional* rates (*u* -1, *d* - 1) is the same as the expected value of the factors (*u*, *d*) minus 1. For proportional rates $\mu = 1 + \theta$ (with θ not in logs).
- Conclusion 4: For the single period, the *proportional* spread is neither affected by the linear shift so that $\tau = \sigma$.
- Conclusion 5: Even though the superior model is in terms of logs the proportional rates remain instructive for reference and human instruction.

### xx.x.7 Risk in the lognormal distribution

xx.x.7.1 The basic relations

We can write $\mu = \mathrm{E}[x] = \mathrm{E}[0 < x < 1] + \mathrm{E}[x \geq 1] = \alpha + \beta$ with $\alpha = \mathrm{E}[0 < x < 1]$ called "the expected outcome of nonpositive growth" or "expected downness" and $\beta = \mathrm{E}[x > 1]$ called "probable profit". With $p = \mathrm{Pr}[x \geq 1]$ we normalize the outcomes to get the (conditional) loss or down and profit or up factors. This gives:

- Profit or up factor $\upsilon = \beta / p = \mathrm{E}[x \mid x > 1]$ with relative profit rate $P = \upsilon - 1$
- Loss or down factor $\delta = \alpha / (1 - p) = \mathrm{E}[x \mid 0 < x < 1]$ with relative down rate $(-L) = \delta - 1$
- Expected value $\mu = \alpha + \beta = p\, \upsilon + (1 - p)\, \delta$
- Expected rate of return $\mu - 1 = p\, P + (1 - p)\, (-L)$
- If $L \geq 0$ then there is a standard Prospect$[P, -L, p]$

It is instructive to consider the plot where the factors are shifted to the left with 1, our capital at the start.

- The lognormal distribution with parameters $\mu = \mathrm{E}[x] = 1$ and $\tau = 0.3$, shifted one unit to the left.

Plot[PDF[ENMT[1, 0.3], x + 1], {x, −1, 2}, PlotRange → All]
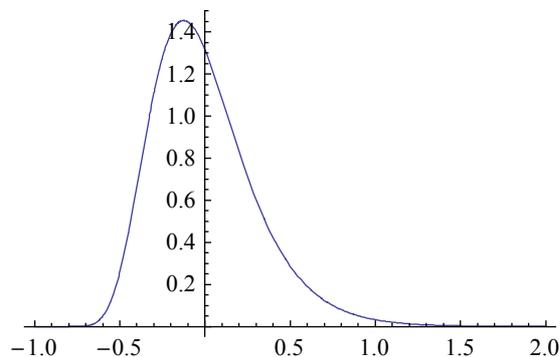
xx.x.7.2 Definition and deduction of risk

We have a proper risk that considers negative values as ρ = - E[-1 < x < 0 | *shifted*] and here the minus sign turns up. When we shift all μ, α and β with one unit to the left then we might expect μ - 1 = (α - 1) + (β - 1) but of course that gives one -1 too much. The proper expectation is different, with ρ = -E[-1 < x < 0 | *shifted*] = $\int_0^1$ (1 - x) pdf[x | *unshifted*] dx = (1 - p) - E[0 < x < 1], where for example -2/3 in the shifted graph must have the comparable density of 1/3 in the original graph. Thus the proper risk for the rates ρ = (1 - p) - α, i.e. the probability of having a downfactor corrected for its benefit. We can find the same correction when we consider the above relation for the rates, μ - 1 = p P - (1 - p) L.

The integral of the density over the interval [0, 1] gives 1 - *p*. *T*his must dominate the integral that is weighted with values smaller than 1, which is α. Thus 1 - *p* ≥ α so that ρ ≥ 0 and the higher the risk the worse it is. This implies that -*L* = δ - 1 = α / (1 - *p*) - 1 ≤ 0 (multiply both sides with 1 - *p* ≥ 0). Thus, in the graph above the value δ - 1 is to the left of 0 and the value υ - 1 to the right.

When we apply the standard definition on risk for Prospect[*P*, *-L*, *p*] then this gives *risk* = (1 - *p*) *L* = (1 - *p*)(1 - δ) = (1 - *p*) - α = ρ, which is consistent.

- These are examples of α , 1 - *p*, ρ and δ.

  {EVZeroToOne[1, 0.3], ENMTLossPr[1, 0.3],
  ENMTRisk[1, 0.3], ENMTDownFactor[1, 0.3]}

  {0.440382, 0.559618, 0.119235, 0.786934}

xx.x.7.3 Numerical example and the ERisket object

Mathematically, when μ = 1 then α = E[0 < x < 1] = *p*. This is a surprising property of the lognormal distribution. It does fit of course with the notion that 1 - *p* ≥ α, since the lognormal is an asymmetric distribution around μ = 1. When interpreting the figures below it is useful to be aware of it. Also, we are considering the normal Prospect here and not the EProspect discussed below.

- When μ = 1 then α = *p*.

  EVZeroToOne[1, τ] == ENMTProfitPr[1, τ]

  True

This is the Prospect[v - 1, δ - 1, p]. This is what routine ENMTToProspect[μ, τ] does.

q = Prospect[ENMTProfitFactor [μ, τ] − 1,
ENMTDownFactor [μ, τ] − 1, ENMTProfitPr [μ, τ]]

$$\text{Prospect}\left(\frac{2\left(\mu - \frac{1}{2}\,\mu\,\text{erfc}\left(\frac{\tau\left(\frac{\log(\mu)}{\tau^2}+\frac{1}{2}\right)}{\sqrt{2}}\right)\right)}{\text{erfc}\left(\frac{\tau^2-2\log(\mu)}{2\sqrt{2}\,\tau}\right)} - 1,\ \frac{\mu\,\text{erfc}\left(\frac{\tau\left(\frac{\log(\mu)}{\tau^2}+\frac{1}{2}\right)}{\sqrt{2}}\right)}{2\left(1-\frac{1}{2}\,\text{erfc}\left(\frac{\tau^2-2\log(\mu)}{2\sqrt{2}\,\tau}\right)\right)} - 1,\ \frac{1}{2}\,\text{erfc}\left(\frac{\tau^2-2\log(\mu)}{2\sqrt{2}\,\tau}\right)\right)$$

ProspectEV[q] // Simplify

$\mu - 1$

ENMTToProspect [μ, τ]	gives Prospect[$v - 1$, $\delta - 1$, p] such that $\delta = \alpha\,/\,(1 - p) = \text{E}[x \mid 0 < x < 1]$ and $v = \beta\,/\,p = \text{E}[x \mid x > 1]$

Use ToBaselineEProspect[μ, τ] to get the factors. Not to be confused with ToLogEProspect[μ, τ] that gives other factors.

- Above parameter values applied.

qn = q /. {$\mu \to$ 1.0, $\tau \to$ 0.3}

Prospect(0.270754, −0.213066, 0.440382)

- The standard Risket gives the expected value, the spread, the risk and the 1 - p, of course all defined with respect to the prospect and not our original problem. The spread is a bit low compared to the original $\tau = 0.30$. The risk or probable loss is a down rate of growth of 11.9%.

ToRisket[qn] // Chop

Risket(0, 0.240184, 0.119235, 0.559618)

- The ERisket object directly follows from μ and τ and contains μ, τ, ρ, 1 - p, v and δ. Note that v and the δ can be calculated from the first four items using $\alpha = 1 - p - \rho$ but they are included for later comparisons.

ToERisket[1, 0.3]

ERisket(Baseline, 1, 0.3, 0.119235, 0.559618, 1.27075, 0.786934)

ENMTRisk $[\mu, \tau]$ the $\rho = 1 -$
ExpectedValue$[0 < x < 1]$ of ExpNormalMuTau $[\mu, \tau]$

ENMTLoss $[\mu, \tau]$ gives $1 - \rho / (1 - p)$ of LogNormalDistribution $[\text{Log}[\mu] - \tau^{\wedge}2/2, \tau]$

Similar routines are ENMTLossFactor, ENMTLossPr, ENMTProfit, ENMTProfitFactor, ENMTProfitPr, ENMTProbableProfit.

ERisket [Baseline,   is an object with expected value of factor μ,
$\mu, \tau, \rho, 1 - p, \upsilon, \delta]$ volatility τ for the Log[factors], ρ risk,
                    probability p of profit, up υ and down δ factors,
                    while these parameters are related in
                    multiplicative fashion. Note that $\alpha = 1 - p - \rho$

ToERisket $[\mu, \tau]$ gives the ERisket object. Option Method controls the kind, default Base

xx.x.7.4 Relation to the discussion below

Below, we will meet the interpretation of an EProspect in terms of a repeated game. This uses parameters or rather estimates θ = EProspectTheta and τ = EProspectTau. Above Prospect can be turned into an EProspect by inserting 1 and then we can give the same estimate. For the overview it is clearest to show this at this point and not at a later point. There is reason to affix more confidence in μ and p than in the other parameters. Of course, in a repeated game the factors may have another interpretation.

- The EProspectTheta and EProspectTau estimates on EProspect$[\upsilon, \delta, p]$ do
  not recover the true θ and τ.

  BaselineEstThetaTau [1.05, .12]

  {ENMT → {0.0415902, 0.12}, Est → {0.0444319, 0.0942564}}

ToBaselineEProspect   gives EProspect$[\upsilon, \delta, p]$
$[\mu, \tau]$

BaselineEstThetaTau $[\mu, \tau]$ returns the θ and τ from ENMT and the implied estimate

Not to be confused with ToLogEProspect$[\mu, \tau]$ that gives other factors that would conform to those estimates.

Since the EProspect can contain all kinds of factors we will use the label "Baseline" to refer to the Prospect context and the label "Log" to the estimates provided by θ =

EProspectTheta and τ = EProspectTau.

xx.x.7.5 Conclusions

Thus we have succeeded in one of our goals, i.e. to create a prospect such that it summarizes the long run situation.

- Conclusion 1: For every lognormal distribution there is a Prospect[*P, -L, p*] that summarizes key long run characteristics of the process, and such that μ - 1 = *p P* - (1 - *p*) *L*. A drawback is that the spread implied by the prospect differs a bit from the τ and differs also from the $\sigma[\mu, \tau]$ implied by that lognormal distribution. But when the emphasis is on the risk then this is a minor point.

- Conclusion 2: The lognormal and its parameters represent some ephemeral long run that need not materialize in an actual run or simulation. The prospect that summarizes it can exist in the same way, i.e. without a definition of a unit period. A public relations problem for the prospect is that it also might represent a wheel of fortune that is actually applied with some unit period. Then E[*S*] = *p* υ + (1 - *p*) δ is defined for long run process *A* and its prospect may be used in process *B* with $\mu[1]$ = *p* υ + (1 - *p*) δ. But there need be no other connection between *A* and *B*.

- Conclusion 3: The development of risk is one of the influences that help us to focus on the *p* in the multiplicative prospect. When comparing a lognormal outcome with a prospect then it is interesting to keep the odds the same. In a two-parameter world we would have a preference for {μ, p} above {μ, τ}.

- Conclusion 4: There is a bit of a paradox. With the probable downfactor α = (1 - *p*) δ for the *factors* we would naively think that α - 1 would be the probable loss, i.e. risk, on the *rates*. To turn it nonnegative we would tend to use - (α - 1) = 1 - α. The lower α the higher our risk. However, this needs to be corrected with *p*, the probability that this high risk does not occur, making the risk measure a bit lower. This holds for any distribution for nonnegative factors around 1 and is not typical for the lognormal. The measure is consistent with the original notion of risk for prospects and it finds a decent interpretation.

- Conclusion 5: This development of risk used the proportional rates for the level outcome of the lognormal. In this way it conforms to the simple Prospect and is useful for human communication. It is an option to investigate the use of logs as in -E[Log[*S*] < 0] etcetera. Care must be taken that our real focus of interest is μ. The condition Log[μ] > 0 seems more natural and would not lead to different conclusions than already gotten.

xx.x.8 Translating an EProspect into a LogNormal using Mu and Pr

xx.x.8.1 The way back

Let us take Prospect[u -1, d - 1, p] and ask whether we can recover the lognormal behind it. The μ is no problem but then ? We know from the above that the spread is no good indication of the τ. Keeping the odds the same, it is logical to use $\{\mu, p\}$ to find the translation and to generates a proper τ. Implied will be some $\upsilon = E[x \mid x > 1]$ and $\delta = E[x \mid 0 < x < 1]$ which should fit the u and d, otherwise the Prospect isn't really a representation of a lognormal distribution. Note that $\mu = p\ u + (1 - p)\ d$ uses the p too. It will be most useful to work with the levels and not the rates.

xx.x.8.2 No freedom for probability

Let us consider the translation of an EProspect[u, d, p] into a lognormal format. See Luenberger (1998:314) for the related derivation and issue. He suggests that p is a free parameter and then solves the overabundance of degrees of freedom by setting $d = 1 / u$. However p would also be determined by the lognormal parameters. That is, once we provide an adequate interpretation for this p, since we might take some leeway here. However, the common prospect explicitly expresses that $p = \Pr[S \geq 1]$ (where we can neglect the density at 1).

- It is no use to solve for θ since p depends upon both θ and τ.

  Pr[$S \geq 1$] == ENMTProfitPr[μ, τ]

  $$\Pr(S \geq 1) = \frac{1}{2}\ \mathrm{erfc}\left(\frac{\tau^2 - 2\log(\mu)}{2\sqrt{2}\ \tau}\right)$$
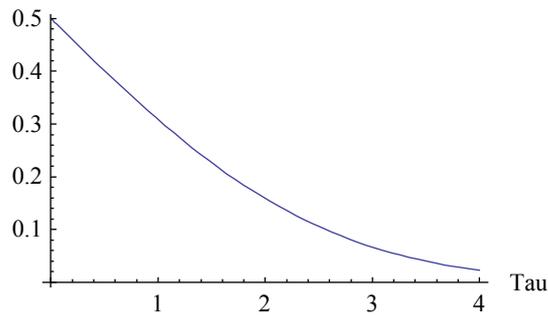
- To the left is the plot for $\mu = 1$. Given that μ is fixed, the limit for larger τ implies that $\Pr[S \geq 1] \to 0$. To the right is the probability density for $\mu = 1$ and τ fixed at 4.

  p1 = Plot[ENMTProfitPr[1, τ] , {τ, .01, 4},
  **   AxesOrigin → {0, 0}, AxesLabel → {Tau, "Pr[S ≥1 | μ = 1] "}];**

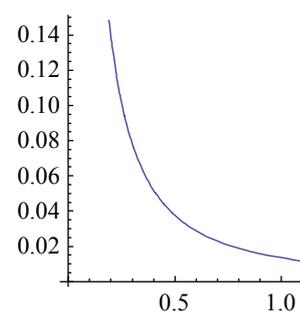 **p2 = Plot[PDF[ENMT[1, 4], x],**
  **   {x, 0, 2}, AxesLabel –> {S, "PDF[S | μ = 1, τ = 4]]"}];**

GraphicsGrid [{{p1, p2}}]

Pr[S ≥1 | μ = 1]   PDF[S | μ = 1, τ = 4]]

- The implied θ and τ follow from $\{\mu, p\}$. In the calculations it appears that we must make a distinction between $\mu < 1$ and otherwise. A situation with structural decline might better be modelled with inverse factors $1 / \mu$.

ENMTTau[μ, p]

If[$\mu < 1$, ENMTTau(1, μ, p), ENMTTau(2, μ, p)]

ENMTTheta[μ, p]

$\log(\mu) - \dfrac{1}{2}$ If[$\mu < 1$, ENMTTau(1, μ, p), ENMTTau(2, μ, p)]^2

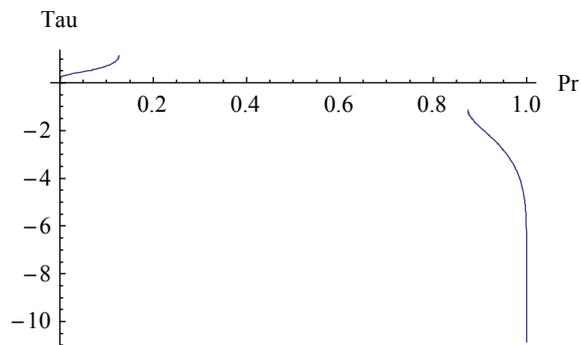- This is the function for τ when $\mu = 1$.

ENMTTau[1, p]

$$\sqrt{2}\left(\sqrt{\operatorname{erf}^{-1}(2\,p - 1)^2} - \operatorname{erf}^{-1}(2\,p - 1)\right)$$

xx.x.8.3 The ranges for Tau

When τ determines p then we have a nice lognormal plot but now we get the inverse. The inverse dependency $\tau[p]$ is not self-evident.

■ You can imagine my mystery when I saw this for the first time.

Plot[ENMTTau[Plot, 1, .52, p], {p, 0, 1},
 **AxesOrigin → {0, 0}, AxesLabel → {Pr, Tau}]**

ENMTTau[μ, p]	gives the value of Tau in the lognormal distribution, while $\mu = E^{(Theta + Tau^2/2)}$

ENMTTau[Plot, μ] shows. ENMTTau[Plot, Show] shows three already selected plots. ENMTTau[1, μ, p] assumes $\mu < 1$. ENMTTau[2, μ, p] for other values. ENMTTau[Plot, 1, μ, p] is required for plotting without the condition on the maximum.

■ This is the explanation. These are prints for $\mu = 0.5$, 1 and 2, where p can determine τ. The equation on the probability generates two solutions. The red lines are for ENMTTau[1,...] and the blue lines are for ENMTTau[2,...]. The first is mostly negative but for a relevant section positive, where it also generates a smaller Tau. Only in the case when $\mu < 1$ there is a maximal value for both p and τ. In this case, and only in this case, an EProspect may contain a p that is too high to generate a proper τ.

ENMTTau[Plot, Show]
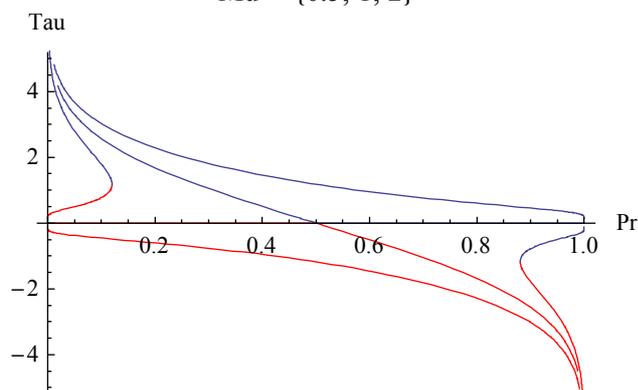
It must be mentioned that there are competing ways to translate an EProspect into a

lognormal. Let us baptize the present method as the Mean manner. Below we shall meet the "Log" manner. Currently we can test an EProspect on the Mean condition.

- This prospect fits the condition of a lognormal distribution.

  q = EProspect[1.05, 0.97, 0.60];

  LogNormalQ[Mean, q]

  True

`LogNormalQ[` `Mean	Log,` `q_EProspect]`	checks whether the EProspect conforms to the assumptions of a lognormal distribution. This is a limited test on Pr. With choice Mean then $\{\mu, p\}$ are used. With choice Log the $\hat{\theta}$ and $\hat{\tau}$ from the prospect are used

When using Log: The EProspectTheta[q] and EProspectTau[q] are substituted again into LogNormalProfitPr[theta, tau].
Returned is Pr == impliedPr, which may remain unevaluated, on purpose. See then also Results[LogNormalQ, Theta].

- Thus it is also possible to create the appropriate μ and τ. This returns to the lognormal parameters. By default the result is into Hold so that we can check on μ and τ. When we release the Hold then we get the θ and τ parameters.

  ToENMT[Mean, q]

  Hold[ExpNormalMuTau][1.018, 0.0626665]

  % // ReleaseHold

  LogNormalDistribution[0.0158764, 0.0626665]

`ToENMT[Mean,` `q_EProspect]`	uses parameters μ and p of the prospect and returns Hold[ENMT][μ, τ] or LogNormalDistribution[Log[μ] − τ^2/2, τ]
`ToENMT[Log, q_EProspect]`	uses implied parameters $\hat{\theta}$ and $\hat{\tau}$ of the prospect and return

Option Hold → True (default) controls the latter switch.

xx.x.8.4 When $\mu < 1$

Let us look deeper into the state of decline. When $\mu < 1$ then the $p[\mu, \tau]$ appears to have a maximum. This means that the p in the EProspect may overstate its case so that it cannot be represented by a lognormal process. The lognormal remains a distribution with limitations - though we might study $1 / \mu$. Let us find the maximum p.

- This gives the maximand, the normal equation, its solution for $\mu \geq 1$ (false, no maximum) and $\mu < 1$ (feasible).

(func = ENMTProfitPr[μ, τ]) $\rightarrow$ (eqs = D[func, τ] == 0)

$$\frac{1}{2} \operatorname{erfc}\left(\frac{\tau^2 - 2\log(\mu)}{2\sqrt{2}\,\tau}\right) \rightarrow -\frac{e^{-\frac{\left(\tau^2 - 2\log(\mu)\right)^2}{8\tau^2}}\left(\frac{1}{\sqrt{2}} - \frac{\tau^2 - 2\log(\mu)}{2\sqrt{2}\,\tau^2}\right)}{\sqrt{\pi}} = 0$$

Reduce[eqs && τ > 0 && $\mu \geq$ 1, {τ, μ}, Reals]

False

Reduce[eqs && τ > 0 && μ < 1, {τ, μ}, Reals]

$$\tau \neq 0 \bigwedge \tau > 0 \bigwedge \mu = e^{-\frac{\tau^2}{2}}$$

- The value of the maximum at this value of μ (or $\theta = -\tau^2$).

pmax == ENMTProfitPr[mu, Sqrt[$-$2 Log[mu]]] /. LogRule

$$\text{pmax} == \frac{1}{2}\operatorname{erfc}\left(-\frac{\log(\text{mu})}{\sqrt{-\log(\text{mu})}}\right)$$

sol = FullSimplify[%, Assumptions $\rightarrow$ {mu > 0, mu < 1}]

$$2\,\text{pmax} == \operatorname{erfc}\left(\sqrt{-\log(\text{mu})}\right)$$

- This prospect has such a low μ that the condition on the maximum strikes. The θ and τ that are generated on that maximum seem acceptable however.

q = EProspect[1.05, .10, .60];

ToENMT[Mean, q]

ENMTTau::max :

  Mean *0.67* *allows maximal probability* *0.185404*. *Its Tau is taken*

Hold[ExpNormalMuTau][0.67, 0.894961]

% // ReleaseHold

LogNormalDistribution [−0.800955, 0.894961]

NB. The lower probability is not used to adapt the μ. That would lead to a lower μ again etcetera.

Thus $\mu < 1$ can generate no, one or two solutions for inverse $\tau[p]$. We can show that by plotting $p[\mu, \tau]$ for a small value of μ. In the left graph, when we draw a horizontal line at $p = 0.5$ then there is no intersection while a horizontal line at 0.10 gives two points. Of course, we can see the same plot in inverse form in the plot above. By default we take the lower τ, in this case about $\tau = 1$, shown on the right.
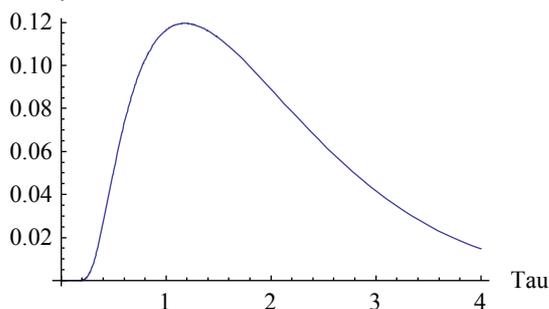
- To the left is the plot for $\mu = 0.5$. Given that μ is fixed, the limit for larger τ implies that Pr[S ≥ 1] → 0. To the right is the probability density for $\mu = 0.5$ and τ fixed at 1.

  p1 = Plot[ENMTProfitPr [0.5, τ] , {τ, .01, 4},
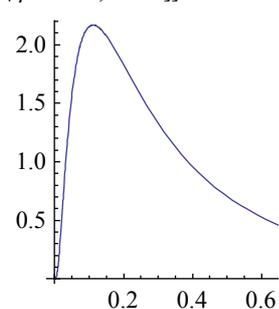 **   AxesOrigin → {0, 0}, AxesLabel → {Tau, "Pr[S ≥1 | μ = 0.5] "}];**

  p2 = Plot[PDF[ENMT[0.5, 1], x],
  **   {x, 0, 1.2}, AxesLabel –> {S, "PDF[S | μ = 0.5, τ = 1]]"}];**

GraphicsGrid [{{p1, p2}}]

xx.x.8.5 Ranges for Theta

When we use $p = 1/2$ then this forces $\theta = 0$ and the lognormal distribution collapses to a one-parameter family. In general however θ follows from μ and τ.

- Solving τ from θ and $p = 1/2$.

    τ **== ENMTTau$\left[\mathbf{Exp}\left[\theta + \tau^2 / 2\right], 1/2\right]$[[3]] // PowerExpand // Simplify**

    $$\sqrt{\tau^2 + 2\,\theta} = \tau$$

- This is a prospect with a $\theta \neq 0$.

    q = EProspect[1.5, 0.77, 0.86];

    ToENMT[Mean, q] // ReleaseHold

    LogNormalDistribution[0.297087, 0.274999]

    LogNormalQ[Mean, q]

    True

xx.x.8.6 Checking on the factors

Above, we only checked whether the prospect *conformed* to a lognormal distribution by checking whether $p = p[\mu, \tau]$ but we did not check on whether it *fitted* in terms of the factors. There seems to be a degree of freedom on *u* and *d*. There is a line $\mu = p\,u + (1 - p)\,d$ where every $0 < d < 1$ generates another *u*. However, in this case the assumption is that $u = \upsilon$ such that $p\,\upsilon = \beta = E[0 < x < 1]$ and similarly for *d*. We can recover the implied factors from the ERisket object and do a test, and put this procedure into a routine FitQ.

- When the prospect has been created from a lognormal then we can recover the parameters of that lognormal.

    q = ToBaselineEProspect[1.05, .3]

    EProspect(1.29466, 0.800354, 0.50504)

    LogNormalFitQ[Mean, q]

    True

ToENMT[Mean, q]

Hold[ExpNormalMuTau][1.05, 0.3]

- An arbitrary EProspect however may contain factors that correspond to a
  lognormal but that do not fit the assumptions of that lognormal with the
  same μ and p.

q = EProspect[1.5, 0.77, 0.86];

LogNormalFitQ[Mean, q]

False

LogNormalQ[Mean, q]

True

ToENMT[Mean, q]

Hold[ExpNormalMuTau][1.3978, 0.274999]

LogNormalFitQ[Mean | Log ?, $q\_$ EProspect, n: Automatic]	checks whether the EProspect fits exactly to the lognormal distribution with the same factors. With choice Mean then $\{\mu, p\}$ are used and the implied factors from ToENMT[Mean, q]. With choice Log the $\hat{\theta}$ and $\hat{\tau}$ from the prospect are used and ????

The routine first calls LogNormalQ. Use n to round to n digits. The routine uses Equal, see \$MaxExtraPrecision. When using Log:

xx.x.8.7 Turning to a repeated game

Let us consider a Prospect[υ -1, δ - 1, p] representing a long term state of choice. We can turn this now into a lognormal distribution with the appropriate parameters. Suppose that we now want to do a repeated game. There are at least two possibilities. The first is to use EProspect[υ, δ, p]. It is not guaranteed that this recovers the long run state since the spread will be off. However, with the lognormal distribution we can find another projection that fits the τ. This is the subject of the next subsections.

xx.x.8.8 Conclusions

- Conclusion 1: Any multiplicative prospect can be turned into a lognormal distribution such that $\mu = E[S] = p\,u + (1 - p)\,d$ and, except for some degenerate cases when $\mu < 1$, in such a way that $p = \Pr[S \geq 1 \mid \mu, \tau]$ so that the odds are the same. Binary prospects thus can be used to represent long term choice situations.

- Conclusion 2: The τ follows from $\{\mu, p\}$ and θ subsequently from $\{\mu, \tau\}$. Via the lognormal process the θ and τ generate the μ and p at the end of the period. Degenerate cases derive not from this deduction itself but from properties of the lognormal distribution.

- Conclusion 3: This approach fits with our knowledge that the σ of a prospect is no good indicator of the τ when the prospect has been created from a lognormal in the "baseline" fashion.

- Conclusion 4: When a lognormal is projected into an EProspect then the μ and τ can normally be fully recovered from the μ and p. There could be projection methods that change μ or p but it is not clear what the advantage of such projections would be.

- Conclusion 5: EProspects not generated in this way will conform to some lognormal (except for degenerate cases) but there will be differences on the factors.

xx.x.9 An alternative course to find a closer fit between τ and σ

xx.x.9.1 In general

As said, the long run has already been essentially solved with the $\{\mu, p\}$ approach. In a two-parameter family there is little room to choose. There may be other cases when the σ of an EProspect is important and would be regarded as indicative for the τ of the lognormal. The discussion on the σ and τ makes most sense in the context of repeated games. With above methods we do not have a good fit so let us try to find a better one.

If σ and τ are the only criterion then it would be straightforward to concentrate on them. This however may obscure the problem. Volatility is not a complete measure of risk. It will be useful to extend the horizon so let us also consider risk in a repeated game.

xx.x.9.2 The risk of a binary EProspect

The format of the Prospect inspires us to define a multiplicative risk.

- The basic binary Prospect format.

  q = $Prospect[]; {q, μ → ProspectEV[q], ρ → Risk[q, Position → True]}

  {Prospect(Profit, −Loss, Pr), μ → Pr Profit − Loss (1 − Pr), ρ → Loss (1 − Pr)}

For the lognormal distribution we start with $\text{Log}[\mu] = \theta + \tau^2 / 2$ and note that the term on the volatility is nonnegative.

- $\text{Log}[\mu] = \theta + \tau^2 / 2 = p\, \text{Log}[u] + (1 - p)\, \text{Log}[d] + \tau^2 / 2 = \{p\, \text{Log}[u] + \tau^2 / 2\}\ -$
  $(1 - p) \{ -\text{Log}[d] \}$
- $\text{LossAnalog} = \text{Log}[1 + f^*] = -\text{Log}[d]$
- $\text{RiskAnalog} = (1 - p)\, \text{LossAnalog} = (1 - p)\, \text{Log}[1 + f^*]$
- $\text{ProductRisk} = E^{\wedge}\text{RiskAnalog} = \Pi = (1 / d)^{1 - p} = d^{p - 1}$
- Hence $\mu = \text{Exp}[\tau^2 / 2]\, u^p / \Pi$ or $\mu\, \Pi = \text{Exp}[\tau^2 / 2]\, u^p$ or   Median $\Pi = u^p$

This fits with the notion that higher risk is unfavourable. The higher the risk the more we fall behind in growth. If we hadn't had the risk then our result would have been higher. When we multiply the ProductRisk with the expected value then we get the untainted outcome of all the wins in the run inclusive the up-factor of volatility.

- The $EProspect[] is the formal object.

  $EProspect[]

  EProspect(ProfitFactor, LossFactor, Pr)

  ProductRisk[$EProspect[]]

  $\text{LossFactor}^{\text{Pr}-1}$

- In practice we tend to interprete 1 - 0.05 as a 5% loss. We should actually
  look at 1 / 1.05 = 1.05263. NB. We can check this by setting $p = 0$.

  { ProductRisk[EProspect[1, .95, 0]], ProductRisk[EProspect[1, 1 / 1.05, 0]]}

  {1.05263, 1.05}

We can summarize the situation. The following equations do not use logs to allow easier translation to Prospect[Profit, -Loss, Pr].

RiskEquations [$EProspect]

$$\{\text{Loss} = \text{DownFactor} - 1, \text{Profit} = \text{ProfitFactor} - 1, \text{Median} = \text{DownFactor}^{1-\text{Pr}} \text{ProfitFactor}^{\text{Pr}},$$

$$\text{ProductRisk} = \text{DownFactor}^{\text{Pr}-1}, \text{DownFactor} = \frac{1}{\text{FailureRate} + 1}\}$$

- A 50 / 50 chance of winning 5% and losing 3% seems like a good deal. The expected value is higher than 1 though perhaps not as much as we might have expected. The product risk actually is 1.5%.

q = EProspect[1.05, 0.97, 0.50];

{SinglePeriodEV[q], ProspectEV[q], ProductRisk[q]}

{1.01, 1.01, 1.01535}

`ProductRisk [ e_EProspect ]`	gives the product risk of the multiplicative prospect

This differs from normal Risk in that there is no Prospect[Profit, -Loss, Pr] but an EProspect[ProfitFactor, LossFactor, Pr] with a different calculation of the expected value, see ProspectEV[$EProspect[]]

xx.x.9.3 For the lognormal

Currently we will discuss another kind of projection from a lognormal distribution with parameters $\{\mu, \tau\}$ into an EProspect[u, d, p]. Let us call this projection *H*. This will create another downfactor and profitfactor. These names are a bit hybrid, by the way. The term upfactor need not be used since it really concerns profit so we stick to the name profitfactor. The term lossfactor however is less inviting since loss would concern the negative area of the shifted lognormal.

- DownFactor $d = H[\mu, \tau, low]$
- ProfitFactor $u = H[\mu, \tau, high]$
- Median $= u^p d^{1-p}$
- ProductRisk $\Pi = (1 / d)^{1-p} = d^{p-1}$
- Expected value $\mu = \alpha + \beta = p\, \upsilon + (1 - p)\, \delta = u^p d^{1-p} \text{Exp}[\tau^2 / 2]$ (with in general $u \neq \upsilon$ and $d \neq \delta$)

- These are the summary equations for the four factors without their explicit solution. Here α is the expected value from 0 to 1 and the υ and δ from above are labeled ENMT so that we have a new ProfitFactor and DownFactor.

RiskEquations[ENMT]

$$\Big\{\text{Mean} = \text{EVZeroToOne} + \text{ProbableProfit}, \text{ENMTDownFactor} = \frac{\text{EVZeroToOne}}{1 - \text{Pr}},$$

$$\text{Loss} = 1 - \text{ENMTDownFactor}, \text{ENMTProfitFactor} = \frac{\text{ProbableProfit}}{\text{Pr}},$$

$$\text{Profit} = \text{ENMTProfitFactor} - 1, \text{Median} = \text{DownFactor}^{1-\text{Pr}} \text{ProfitFactor}^{\text{Pr}},$$

$$\text{ProductRisk} = \text{DownFactor}^{\text{Pr}-1}, \text{DownFactor} = \frac{1}{\text{FailureRate} + 1}\Big\}$$

xx.x.10 LogNormal to an EProspect and vice versa, via Theta and Tau

xx.x.10.1 Key assumption on Tau

The key assumption is that the spread of above logarithmic (non-E) Prospect provides the τ of a lognormal distribution. When we combine this with *median* $= e^{\theta} = u^p d^{1-p}$ or $\theta = p$ Log[u] + (1 - p) Log[d] then we have two equations for two unknowns, and expressed in the two parameters θ and τ of the lognormal distribution.

- The logarithmic Prospect generates a Tau that depends upon (a) the geometric mean of the probabilities, *times* (b) the absolute difference of the logs of the factors. Since $u > d$, it is only the difference and we can write this as Log[u / d].

qlog = ToUtility[Prospect @@ \$EProspect[]] /. Utility → Log /. Pr → p

Prospect(log(ProfitFactor), log(LossFactor), p)

Tau == (Spread[qlog] // Simplify)

$$\text{Tau} = \sqrt{-(p-1)\, p\, (\log(\text{LossFactor}) - \log(\text{ProfitFactor}))^2}$$

Tau == EProspectTau[\$EProspect[]]

$$\text{Tau} = \sqrt{(1 - \text{Pr})\,\text{Pr}}\;\log\!\left(\frac{\text{ProfitFactor}}{\text{LossFactor}}\right)$$

PM. Again, Log[x] $\approx$ x - 1 when x itself is around 1. In that case this τ will not differ much from the σ of the linear Prospect, either in x or in x - 1.

PM. The estimate of the median fits what we know about the repeated game. We can include μ that implies another estimate for τ. But we already know that θ and τ will deviate from the long run state. There is little need now to extend on the range of possible estimates. We pursue the main line now which is to take the spread of the logs.

`EProspectTau[`	gives the implied value of Tau (using the logs of the factors)
$q\_$`EProspect]`	

xx.x.10.2 No freedom for probability

Subsection xx.x.8.2 already clarified that $p = p[\theta, \tau]$. We cannot use θ and τ independently from p or with some p in the estimate without checking for consistency. This statement will be repeated some times in this discussion but it is useful to do so here because in the former sections p was the independent variable.

xx.x.10.3 From the lognormal to the EProspect

The best approach now is to assume the lognormal and project it into the prospect. Let us assume θ and τ, determine implied $p = \Pr[S \geq 1 \mid \theta, \tau]$, assume that we have done so with one of our routines, and solve for the factors u and d:

- $\Delta = \text{Log}[u / d] = \tau / \sqrt{(1 - p)\, p}$ from $\tau = \sqrt{(1 - p)\, p}\ \text{Log}[u / d]$
- $\theta = p\, \text{Log}[u] + (1 - p)\, \text{Log}[d] = p\, \Delta + \text{Log}[d]$
- $\text{Log}[d] = \theta - p\, \Delta$
- $\text{Log}[u] = \Delta + \text{Log}[d] = \Delta + (\theta - p\, \Delta) = \theta + (1 - p)\, \Delta$

Some terms can be simplified a bit:

- $p\, \Delta = \tau / \sqrt{1 / p - 1}$
- $(1 - p)\, \Delta = (1 / p - 1)\ p\, \Delta = (1 / p - 1) . \ \tau / \sqrt{1 / p - 1}\ = \tau\, \sqrt{1 / p - 1}$

Hence:

- $d = \text{Exp}[\theta - \tau / \sqrt{1 / p - 1}\,] = e^{\theta}\, \text{Exp}[-\tau / \sqrt{1 / p - 1}\,]$
- $u = \text{Exp}[\theta + \tau\, \sqrt{1 / p - 1}\,] = e^{\theta}\, \text{Exp}[\tau\, \sqrt{1 / p - 1}\,]$

It appears that the median provides a common scalling factor in u and d. Also $d \neq 1 / u$ except for special cases, see the Luenberger (1998) assumption. The following

applications use a formal p for presentation only. It must be understood that this p is actually a function of the other parameters too so that it does not represent an independent choice.

- A projection from the lognormal into an EProspect. NB. p is not independent.

    q = LogNormalToEProspect [θ, τ, p]

$$\text{EProspect}\left(e^{\theta + \sqrt{\frac{1}{p} - 1}\ \tau},\ e^{\theta - \frac{\tau}{\sqrt{\frac{1}{p} - 1}}},\ p \right)$$

- Using $\{\mu, \tau\}$ makes the proportionality a bit more explicit, though τ remains in the exponent. NB. p is not independent.

    q = ToLogEProspect[μ, τ, p]

$$\text{EProspect}\left(e^{\sqrt{\frac{1}{p} - 1}\ \tau - \frac{\tau^2}{2}}\ \mu,\ e^{-\frac{\tau^2}{2} - \frac{\tau}{\sqrt{\frac{1}{p} - 1}}}\ \mu,\ p \right)$$

- This is a numerical projection.

    qOK = ToLogEProspect[1.12, 0.3]

    EProspect(1.37488, 0.747058, 0.590084)

`ToLogEProspect [ ` μ`, `τ`]`	gives q = EProspect[u, d, p] that fits ToENMT[Mean | Log, q]

Not to be confused with ENMTToProspect[μ, τ] or ToBaselineEProspect[μ, τ]. ToLogEProspect[μ, τ, p] does not use p = ENMTProfitPr[μ, τ] and can be used for display. Be careful in using numerical values here because the prospect may not satisfy the lognormal distribution. See LogNormalQ. Another version is LogNormalToEProspect[θ, τ] that assumes that the transformation is to a "Log" EProspect.

- The produces a series of criteria based upon only two parameters μ and τ. When $\mu = 1$ then $\alpha = p$ so that expected downness is the same as the probability of profit, so that $\rho = 1 - 2\,p$.

    res = ENMTImpliedFactors [1, 0.3]

    {Mean → 1, Tau → 0.3, Theta → −0.045, Median → 0.955997, Spread → 0.306878,
      Pr → 0.440382, Profit → 0.270754, ProbableProfit → 0.559618, Loss → 0.213066,
      Risk → 0.119235, ENMTProfitFactor → 1.27075, ProfitFactor → 1.34069,
      ENMTDownFactor → 0.786934, DownFactor → 0.73262, Dif → 0.608068,
      Ratio → 1.82999, ProductRisk → 1.19019, FailureRate → 0.364964}

This selects key criteria and can be compared to the ERisket with the Baseline method.

ToERisket[1, 0.3, Method → Log]

ERisket(Log, 1, 0.3, 1.19019, 0.559618, 1.34069, 0.73262)

ENMTImpliedFactors [μ, τ]

      solves the DownFactor and ProfitFactor using
      ToLogEProspect and generates additional statistics

Dif = *u - d* and Ratio = *u / d*.

xx.x.10.4 Conversely, from an EProspect to a lognormal distribution

For the converse, it is straightforward to calculate $\theta = p \, \text{Log}[u] + (1 - p) \, \text{Log}[d]$ and $\tau = \sqrt{(1 - p) \, p} \; \text{Log}[u \, / \, d]$ while using the p that is available in the prospect. There will be an implied $p[\theta, \tau]$ though. We include a test on the implied p and let the routine generate a warning message when the prospect p differs from the implied p.

- This returns to the lognormal parameters. By default the result is into Hold so that we can check on μ and τ. When we release the Hold then we get the θ and τ parameters. Of course, the prospect generated in the former subsection gives no error message since the p already has been calculated on the parameters.

ToENMT[Log, qOK]

Hold[ExpNormalMuTau][1.12, 0.3]

ReleaseHold[%]

LogNormalDistribution [0.0683287, 0.3]

This establishes the "Log" translation ExpNormalMuTau → EProspect → ExpNormalMuTau. The reason why this works is that the prospect has been created to satisfy a condition on the u and d and p terms. This does not mean that *any* EProspect will fit the lognormal distribution perfectly. The lognormal distribution is essentially a two-parameter distribution.

- The following example shows that not every prospect generates a lognormal distribution such that the prospect p equals $\Pr[S \geq 1 \mid \theta, \tau]$.

  q = EProspect[1.05, .97, .50];

  ToENMT[Log, q]

  LogNormalQ ::neq :

  EProspect Pr 0.5 not equal to Pr 0.591462 implied by Theta and Tau

  Hold[ExpNormalMuTau][1.01, 0.0396247]

- The formal \$EProspect gives us the condition that Pr has to satisfy. Let us use Ratio = u / d.

  cond = LogNormalQ[Log, \$EProspect[] /. ProfitFactor → Ratio LossFactor];

  cond2 = FullSimplify[cond, Assumptions →
  {Pr > 0, Pr < 1, LossFactor > 0, ProfitFactor > LossFactor}]

$$2\,\mathrm{Pr} = \mathrm{erfc}\left(-\frac{\log(\text{LossFactor}) + \mathrm{Pr}\,\log(\text{Ratio})}{\sqrt{2}\,\sqrt{-(\mathrm{Pr}-1)\,\mathrm{Pr}\,\log(\text{Ratio})}}\right)$$

xx.x.10.5 Conclusions

These conclusions are mainly for the context of the repeated game:

- Conclusion 1: For every lognormal distribution there is a $\{\theta, \tau\}$ projection into a multiplicative EProspect[u, d, p] where $p = \Pr[S \geq 1]$. This has an unambiguous interpretation for the single period but for repeated games we must allow that the path traverses below 1 while only ending at or above 1.

- Conclusion 2: For every multiplicative EProspect[u, d, p] there is a θ and a τ so that we can construct a lognormal distribution. But the implied $\hat{p} = \Pr[S \geq 1 \mid \mu, \tau]$ may not fit the p of the prospect.

- Conclusion 3: This has an unambiguous interpretation for the single period: it contradicts the idea that any prospect would end up into a lognormal distribution. With an arbitrary wheel of fortune we *apparently* (unless other evidence comes in) cannot use the one period parameters to model the long run behaviour, unless the parameters already fit a lognormal distribution.

- Conclusion 4: For repeated games we would have to *choose*: (a) The limit distribution might be another than the lognormal though there can be a good approximation. (b) Or, the p in the one period prospect only gives the period-to-

period probability but does not give the long run interpretation. The implied p is the true long run probability. (The message by LogNormalQ is only a warning and not an error.)

xx.x.11 Comparison of Prospect and EProspect for the lognormal

xx.x.11.1 A systematic error on the main variable

Above conclusions run counter to the situation when the prospect is used as a summary of the long run situation. Indeed, above approach generates a systematic difference on the mean. Thus $\mu[1]$ differs from $\mu = E[S]$.

- The error is expressed in logs to highlight negative rates of growth. The log[Exp[...]] term is not simplified to keep the terms together.

  ENMTErrorOnMu [EProspect[u, d, p]]

  $$\log(d\,(1-p) + p\,u) - \log\left(e^{\frac{1}{2}\,(1-p)\,p\,\log^2\left(\frac{u}{d}\right) + (1-p)\,\log(d) + p\,\log(u)}\right)$$

- For small parameter values it does not matter much but for larger values it does.

  ENMTErrorOnMu [EProspect[1.05, .97, .5]]

  -2.05353×10^{-7}

  ENMTErrorOnMu [EProspect[1.50, .50, .7]]

  -0.0202895

- This is what happens to the one period expectation $\mu = p\,u + (1 - p)\,d$ when we use the parameters of an established lognormal distribution.

  subst = Thread[{u, d, p} → List @@ qOK]

  $\{u \to 1.37488, d \to 0.747058, p \to 0.590084\}$

  {μ[1] == p u + (1 − p) d, E[S] == ProspectEV[EProspect[u, d, p]]} /. subst

  $\{\mu(1) = 1.11753, e[S] = 1.12\}$

The reasoning is that the lognormal distribution is only a two-parameter family so that adding $\mu = p\,u + (1 - p)\,d$ to the equations causes overdetermination. Still, we would like to use a prospect to represent a long run average situation - and it remains a key notion that the unit period is an arbitrary concept.

xx.x.11.2 Using a degree of freedom

There is a degree of freedom on u and d. Seen from $\{\mu, p\}$ their $\{\theta^*, \tau^*\}$ follow from the formulas of the lognormal distribution. Thus the u and d are not used to the full. There is a line $\mu = p\,u + (1 - p)\,d$ where every $0 < d < 1$ generates another u. There can even be a specific value of d^* that generates the $\{\theta^*, \tau^*\}$ that satisfy the standard formulas. Thus, take $\{\theta^*, \tau^*\}$ as given from $\{\mu, p\}$ and solve for $\{u^*, d^*\}$.
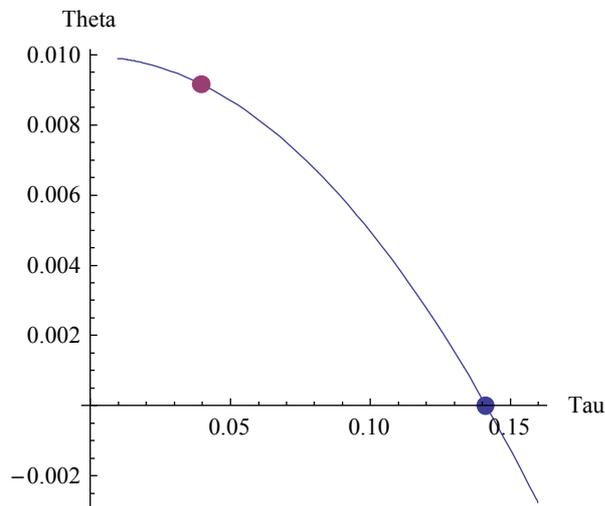
- $\theta^* = p\,\text{Log}[u^*] + (1 - p)\,\text{Log}[d^*]$

- $\tau^* = \sqrt{(1 - p)\,p}\ \text{Log}[u^* / d^*]$

- The purplish dot gives the $\{\theta, \tau\}$ outcomes from the standard formulas for the rates. The blue dot gives the $\{\theta^*, \tau^*\}$ outcomes from using $\{\mu, p\}$. The dots are on the line $\mu = p\,u + (1 - p)\,d$ (in another dimension).

    q = EProspect[1.05, .97, .50];

    ToENMT[Mean, Plot, q, {d, .85, 1}]

The following is a bit convoluted. We collect the μ and p from the prospect, calculate the τ and create a prospect using the Log method. This translates the original prospect q into another prospect q^* with the same $\{\mu, p\}$ but with $\{u^*, d^*\}$ such that use of the rate formulas generate $\{\theta^*, \tau^*\}$ that fit the same original $\{\mu, p\}$.

- We do this in two slightly different ways with a digit of difference.

    q = EProspect[1.05, 0.97, 0.50];

```
mu = SinglePeriodEV[q]; prob = q[[3]];
ToLogEProspect[mu, ENMTTau[mu, prob]]
```

EProspect(1.1515, 0.868429, 0.5)

```
qstar = ToENMT[Mean, Fit, q]
```

EProspect(1.15157, 0.868429, 0.5)

```
SinglePeriodEV[q] == SinglePeriodEV[qstar]
```

True

The issue can be seen as a matter of construction. When we have a longer run outcome with $\{\mu, p\}$ and we want to summarize the result and communicate what it means then we can create two factors u and d such that EProspect[u, d, p] generates the same μ. And we can do it in such a way that the rate formulas on the prospect generate the same $\{\theta, \tau\}$, which some may regard as a requirement for consistency. Also, if the original outcome $\{\mu, p\}$ has been generated by a true lognormal process then the $\{\theta, \tau\}$ are the parameters of that process and then the created EProspect perfectly reflects the process.

xx.x.11.3 Conclusions

- Conclusion 1: Contrary to earlier suggestions, any multiplicative prospect can be turned into a lognormal distribution such that $\mu = E[S] = p\,u + (1 - p)\,d = \mu[1]$, and, except for some degenerate cases, in such a way that $p = \Pr[S \geq 1 \mid \mu, \tau]$. Binary prospects thus can be used to represent long term choice situations and the one period expected value (with arbitrary period) need not differ from the long run expectation.

- Conclusion 2: The θ and τ in the standard assumption are not relevant for this other representation. There are values of θ^* and τ^* that, via the same lognormal process, generate the μ and p at the end of the period. Degenerate cases derive not from this other approach but from properties of the lognormal distribution.

- Conclusion 3: The θ and τ in the standard formulas can still be relevant for their own context. For parameters $\{\mu, p\}$ we can find a $\{u^*, d^*\}$ such that $\{\theta^*, \tau^*\}$ satisfy those standard formulas. When $\{u, d\}$ are already given in the prospect then there is the choice between using standard $\{\theta, \tau\}$ or alternative $\{\mu, p\}$. The choice depends upon the reasons for our exercise. Supporters of the standard approach might be able to live with the idea that the prospect represents the long run outcome and not the single period in the lattice - for which $\{u^*, d^*\}$ would

apply. Simulations with binomial lattices may highlight the differences but we will require criteria to evaluate such results.

xx.x.12 Some considerations for more alternatives on the repeated game

First we used $\{\theta, \tau\}$ in the prospect, the alternative was to use $\{\mu, p\}$, and naturally there are more possibilities and mixtures. In general the situation is EProspect$[\alpha\ u, \beta\ d, \gamma\ p] \sim$ ENMT$[\delta\ \mu, \epsilon\ \tau]$ with sidecondition η $(\mu = p\ u + (1 - p)\ d)$ and for an estimate we must determine where the error should go. Any choice comes with its properties.

- The choice of $\{\theta, \tau\}$ is rather strict and excludes many prospects from the lognormal world unless we allow that the single period p is not the implied p.
- The choice of $\{\mu, p\}$ gives much more freedom but of course at the price of dropping the assumptions on θ and τ.
- The choice for $\{\tau, p\}$ for example would imply that the ratio of the factors remains the same.
- It was a consideration to keep p the same because of the odds implied by the prospect but of course there can be other considerations. Our interpretation of p has been strict and we can imagine a bit more leeway.
- For the one period we might choose EProspect$[\alpha\ u, d, p]$ with a risk-averse idea that the down side should remain the same.
- Alternatively $\beta = 1/\alpha$ for a two-sided effect. For a multiperiod setting we might use a binomial lattice to set some floor on the $\Pr[S \geq \min]$ and only accept those paths that remain in this range.
- In general we will need criteria and the notion of risk might help here.

xx.x.13 Appendix: Dependence on time

The binomial lattice or diffusion process is essentially dependent upon time. In the discrete approximation in Luenberger (1998:315) he systematically includes a time variable. This was mysterious at first since the choice of the period can be reduced to the choice of 1. But for a diffusion process the rate of return indeed is only valid *per unit of time*. If our single period prospect is supposed to provide a summary of a true diffusion process then it only applies for a specific period length, i.e. a period that fits with the data entered. For a shorter horizon the diffusion requires a higher μ and for a longer period it must do with a lower value.

- The formulas are rather huge but still printable.

qtime[t] = ToLogEProspect$\left[1.12 ^\wedge t, .3 \sqrt{t}\right]$ /. LogRule // Simplify

$$\text{EProspect}\left(e^{0.0683287\ t+0.3\ \sqrt{\frac{2}{\text{erfc}\left(-0.161052\ \sqrt{t}\right)}-1}\ \sqrt{t}},\right.$$

$$\left. e^{0.0683287\ t-\frac{0.3\ \sqrt{t}}{\sqrt{\frac{2}{\text{erfc}\left(-0.161052\ \sqrt{t}\right)}-1}}},\ \frac{1}{2}\ \text{erfc}\left(-0.161052\ \sqrt{t}\right)\right)$$

ev[t_] = ProspectEV[qtime[t]]   // PowerExpand // Simplify

$$e^{\frac{t\left(0.113329\ \text{erfc}\left(-0.161052\ \sqrt{t}\right)-0.226657\right)}{\text{erfc}\left(-0.161052\ \sqrt{t}\right)-2.}}$$
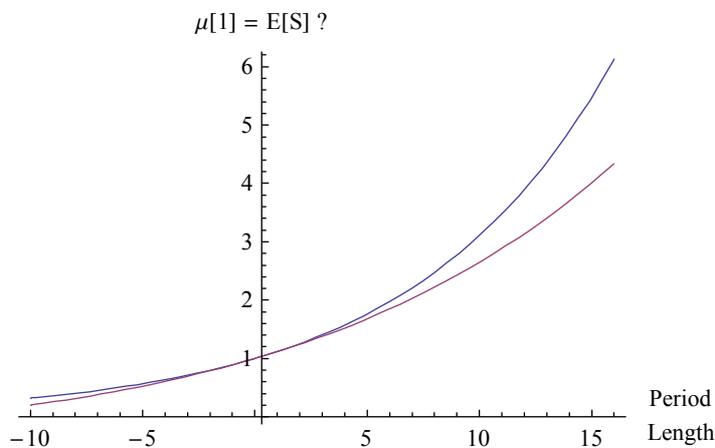
subst = Thread[{u, d, p} → List @@ qtime[t]];

- Let us apply these relations to the single period expectation.

evsingle[t_] = p u + (1 − p) d   /. subst // PowerExpand // Simplify

$$\frac{1}{2}\ e^{0.0683287\ t-\frac{0.3\ \sqrt{t}}{\sqrt{\frac{2}{\text{erfc}\left(-0.161052\ \sqrt{t}\right)}-1}}}\left(\left(-1+e^{\sqrt{\frac{2}{\text{erfc}\left(-0.161052\ \sqrt{t}\right)}-1}\ \text{erfc}\left(-0.161052\ \sqrt{t}\right)}\right)\text{erfc}\left(-0.161052\ \sqrt{t}\right)+2\right)$$

- The lines cross and there is clearly a point of intersection.

Plot[{ev[t], evsingle[t]}, {t, −10, 16}, AxesOrigin → {.3, 0},
  AxesLabel → {"Period\nLength", "$\mu[1]$ = E[S] ?"}]

- This is the length of the period that we would have to take into account.
  Well, FindRoot seems to have a problem but FindMinimum finds a value
  slightly different from zero if we adapt the AccuracyGoal.

FindRoot[ev[t] == evsingle[t], {t, −1, 1.5}] // Chop

$\{t \rightarrow -8.70313 \times 10^{-8} - 9.74447 \times 10^{-8} \, i\}$

FindMinimum [(ev[t] − evsingle[t])^2,
  {t, .4, −3, 3}, AccuracyGoal −> 7] // Chop

$\{0, \{t \rightarrow 3.64223 \times 10^{-7}\}\}$

To a certain extent this discussion is not too convincing. The plot clarifies that there are competing measures $\mu[1]$ and E[S] indeed that collapse to the same value, in fact our original capital 1, when the length of the period gets close to 0. But it is not convincing when we are concerned with some period average and when we like to put for example an annual average into the EProspect for further analysis.

We actually can find an explicit expression for the period length when we assume μ, τ and p given.

- When μ, τ and p are given from the EProspect then we can solve t from the inverse CDF. There is a routine that contains that result. NB. With a note of caution because *Mathematica* warns that inverse functions have been used so that the solution need not be complete.

eqn[t] = p == ENMTProfitPr$\left[\mu\text{^t}, \; \tau \sqrt{t} \; \right]$ /. LogRule // Simplify

$$2\,p = \operatorname{erfc}\left(\frac{\sqrt{t}\,\left(\tau^2 - 2\log(\mu)\right)}{2\,\sqrt{2}\,\tau} \right)$$

t == ENMTTime[μ, τ, p]

$$t = \frac{2\,\tau^2\,\operatorname{erfc}^{-1}(2\,p)^2}{\left(\log(\mu) - \frac{\tau^2}{2}\right)^2}$$

- This works fine when the basic parameters already fit a lognormal distribution.

t == ENMTTime[1.12, 0.3, 0.590084]

$t = 0.999989$

t == ENMTTime$\left[\text{1.12^2}, \; 0.3 \sqrt{2}, \; 0.590084\right]$

$t = 0.499995$

- But it is very sensitive to the parameter values. Changing 0.59 into 0.53 gives a 10% length of the period and 0.50 already gives zero.

t == ENMTTime[1.12, 0.3, 0.53]

$t = 0.109214$

- Conclusion 1: For every lognormal distribution there is a projection into a multiplicative EProspect[u, d, p] where $p = \Pr[S \geq 1]$ and an associated length of the period such that $\mu = E[S] = \mu[1] = p\,u + (1 - p)\,d$. Unfortunately, for many EProspects the length of the period will be close to zero because of the sensitivity of the relations involved. Adjusting for the time period is not the solution to our main problem.

ENMTTime [μ, τ, p]	gives the implied t such that $p ==$ ENMTProfitPr[μ^t, τ Sqrt[t]]

PM. There is the possibility that the EProspect presents a period average. The figures are

a different then. This has not been developed.