Trig rerigged

Trigonometry reconsidered. Measuring angles in 'unit meter around' and using the unit radius functions Xur and Yur

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http://www.dataweb.nl/~cool

Abstract

Didactic issues in trigonometry concern the opaque names of sine and cosine and the cluttering of questions with π or 360 whereas a simple 1 suffices. The solution is to use the 'unit meter around' as the yardstick for angles and to use the Xur and Yur functions for the $\{x, y\}$ co-ordinates on the circle with unit radius. The relevant mathematical constant is $\Theta = 2\pi$ (capital theta, reminiscent of a circle) rather than π and it comes into use much less when we use UMAs instead of radians. The sine and cosine remain relevant for non-oriented angles and the derivative. The common term 'dimensionless' appears to confuse 'no unit of measurement specified' (with a metric, in planimetry and trigonometry) with 'no dimension' (a pure number, in number theory).

Introduction

The current mathematical convention of handling angles in trigonometry has three awkward aspects:

1. The units of measurement of angles are degrees (max 360) or radians (max 2π) or gradians (max 400) instead of a clear 1 (unit unspecified) or 1 meter (unit specified). The conventional measures are ratios and obscure the point that the angle is measured by the length of the arc around a circle. For length there has already been defined a standard, namely the meter, so why not use it again for the circumference? A standard circle with a circumference of 1 meter better clarifies that we are measuring length. The unit of measurement then is 'unit meter around' (UMA). This can be made dimensionless as a 'turn' (as a fraction of that maximal unit length around) or as 'unit of measurement around', where a turn is one unit.
2. Calculations in trigonometry use the mathematical constant π instead of the handier $\Theta = 2\pi$ (capital theta, reminiscent of a circle).
3. The names of the sine and cosine functions do not link up to the already known expressions for the horizontal and vertical axes, i.e. the x and y values. Students have to calculate these x and y values but are not explicitly told to calculate these, i.e. they are told to calculate the sine and cosine which seems as something completely different (or perhaps linked in a manner that is not clear to them). It will be clarifying to use functions Xur and Yur defined on the unit circumference circle and that range on the x and y values of the unit radius circle.

The traditional approach makes mathematical courses more tedious than necessary for understanding angles. The π needlessly clutters the argument in two ways. Students struggle to find the values k * 3.14... on their ruler while it would be more convenient to use 1 for the full circle around. Secondly, if a fraction or multiple of π is to be used at all, it is more convenient to use Θ. The following develops the simplified approach.

There is also the smaller issue that angles are measured counterclockwise and the measurement starts at three o'clock on the dial, while this might be counter-intuitive to students, who might feel that measurement should be clockwise and start at 0 (12 o'clock). When the clock starts ticking at noon then after five minutes the big hand shows an angle of 5 minutes (as measured by that clock). Clocks are indeed basic in our culture and primary schools spent a lot of time to teach children how to interprete clocks. Clocks are also used to indicate spatial directions, such as ‘turn around to 7 o’clock’. However, after some deliberation, one may agree that it is more natural to indicate angles as deviating from the horizontal plane while it is less natural to indicate them as deviating from the vertical axis. To start at {-1, 0} and turning clockwise has the advantage of the horizontal reference and the direction of integration but has the drawback of the negative x-value. Also, there are ample cases where the movement is counterclockwise such as turning a screw loose. Thus it suffices to conceptually disengage clocks and angles as much as possible and only refer to the counterclockwise way of measurement.

After basically completing this article in May, I came across Palais (2001ab), similarly clarifying that 2π is didactically inadequate. Palais introduces the three-legged notation  but this is bound to cause confusion and writing and reading errors and I remain with Θ.

Dimension and metric

It appears that we should distinguish ‘no dimension’ (pure number) from ‘no unit of measurement specified’ (with a metric in space). It also appears that the common term ‘dimensionless’ tends to confuse these two notions, i.e. the term ‘dimensionless’ does not mean quite the same in number theory as in the theory of space. In this section we will use quotes around ‘dimensionless’ to highlight the confusion. In the rest of the paper the term ‘dimensionless’ stands for the context of space. The discussion is a bit complex since notions in number theory tend to derive their names from the theory of space, so that it may be hard to keep distinguishing the two. In all clarity, now, we should distinguish the distance in space that provides a metric in space from the ‘metric’ that may be defined on a set of pure numbers. The ‘metric’ for pure numbers can be based upon a calculation scheme $||z_1 - z_2||$ and the metric in space follows from our experience and conceptualization of space. It is ‘analytical geometry’ to associate the two. Note though that ‘association’ implies that there are two different realms and not necessarily only one. It is a bit amazing that the fundamentals of analytical geometry still clog up the didactics of planimetry and trigonometry but hopefully this section will contribute to more clarity.

Traditionally, an angle is defined as the plane section between two half lines with a same origin. Subsequently this angle is measured, creating the distinction between the angle itself and its measurement. The measures standardly are ratios, e.g. the arc, sine, cosine or tangent, which measures provide ‘dimensionless’ numbers (note the quotes). Notably, when drawing a circle with its center at the same origin of the half lines, the arc measure of the angle is standardly defined as the arc within the plane section divided by the radius of the circle, and this ratio is a 'dimensionless' number. A full circle around would give us an angle of $(2\pi\ r)\ /\ r = 2\pi$ radians. Frequently the 'arc measure of the angle' is replaced by 'angle' itself, and we will tend to do so as well in the present paper.

Note that these 'dimensionless' numbers occur in the human mind and in formulas but not in the actual measurement of distance in reality, e.g. on paper. With M the unit of measurement (say 1 meter) and λ a 'dimensionless' scalar, the measurement would be $x = \lambda\ M$. In more formal dimensional analysis, we get $x = \lambda\ [M]$, where the brackets identify the unit of measurement. Any practical measurement on paper would generate such a dimensional number x. However, since the unit of measurement itself is a phenomenon of the same space (say 1 meter, as the length of a rod in Paris), one may argue that the true measurement is $\lambda = x\ /\ M,$ again a 'dimensionless' number. However, we have to keep in mind (i) what dimension was measured (length) and (ii) with what unit (the meter), so that the measurement would never be only a pure number. For $\lambda = 1$ we would not only have a ratio of 1 but we would be speaking about the measurement of that particular unit of measurement, and it would e.g. have the properties of dimensionality and continuity.

The continuity of all measures derives from their spatial extension. The measures are continuous since when $r = 1$ it appears that the values are given by particular lengths, and lengths are continuous. The sine derives its metric value from the height in the unit circle. The arc derives its metric value from the fact that it is a length, namely part of a unit circle. That the arc can also - note 'also' - be defined as a ratio is secondary and does not make it without dimension but rather with unspecified dimension. Using a ratio is a mathematical simplification, eliminating the need to construct a unit circle, but does not affect the notion that lengths are involved.

These 'dimensionless' numbers or ratios thus cause an epistemological question. Mathematically, we have to distinguish reality from the human mind. It might be that reality is only granular and that continuity is an illusion created in the mind, in the same way as, concerning time, the 'now' is a construct of the mind for the ephemeral border, or actually only logical border, between 'past' and hypothetical 'future'. It is more conventional however to assume that space in reality is continuous and that we create measures in number theory to mimic this property of space. Thus we distinguish crude figures and lengths on paper from the pure figures and 'dimensionless' numbers in the mind. On paper we may take a unit of measurement (say, the rod in Paris) but in the mind there is no place for such a physical object. Thus in planimetry, 'dimensionless' stands for 'no unit of measurement specified'. Indeed, the notion of 'dimensionless' number interpreted as 'no dimension' remains epistemologically dubious when we relate this to the measurement of length, basically on paper and subsequently in the human mind. For, how could it be that these 'no dimension' numbers are nicely ordered and apparently have a distance metric such that e.g. halves are twice as distant as quarters ? Where does the notion of continuity come from ? In practice we assume that space is continuous. Apparently, there is a subtle distinction between 'no dimension' and 'unspecified dimension (unit)'. Apparently, the mind thinks about space with unspecified dimensions and not quite without dimensions. This is similar as drawing a line on paper and arbitrarily affixing 0, 1, 2, ... numbers along it, with the numbers at (approximately) the same distance, and writing down that these are meters while in fact they will be something else, with the true metric defined on the spot. Imagining triangles, circles and line sections in the mind, we must admit that they all have some apparant 'size', albeit 'size in the mind', all in proportion to the other things that we may imagine for comparison. Thus the 'dimensionless' numbers in trigonometry still reflect length, with a space metric, albeit with unspecified unit.

The latter is an important didactical conclusion. Some mathematicians tend to think that trigonometry deals with 'dimensionless' ratios (apparently meaning 'no dimension' as in number theory) and not with length, and they seem to deny that the notion of 'ratio' implicitly has a metric and that this metric is related to the notion of length itself. This present paper suggests to bring the true implicit relation out into the open by explicitly referring to length and the 'unit meter around'. Students in trigonometry then learn to switch between actual length with a specified unit and length without a specified unit (ratios). This would be an advance in clarity compared to the current practice where ratios are defined and where it is suggested that we are not measuring length but merely calculating 'no dimension' numbers as in number theory.

The basic point is that our topic of interest here is space, with its figures and angles. In trigonometry the space metric is a priori, and abstract numbers without dimension (and the number 'metric') support the analysis, but cannot replace that notion of a metric contained in the notion of space. Admittedly, number theory can have its own origins. Possibly we start counting on our fingers and then apply the same technique to spatial distance. But the experience that walking 50 kilometers is more tiring than walking 10 meters, and other experiences with space, need not depend upon counting. Indeed, in number theory we can define a set of numbers without dimension, and there we can define a 'metric' $||z_1 - z_2||$ on those numbers, but this 'metric' is not a metric as in space (real or in the mind). It rather works the other way around. We can take ratios, i.e. express lengths as multiples of a standard unit length, and we can abstract from the space metric to also create such a set of pure numbers, and then what works for space can also be reflected in those numbers.

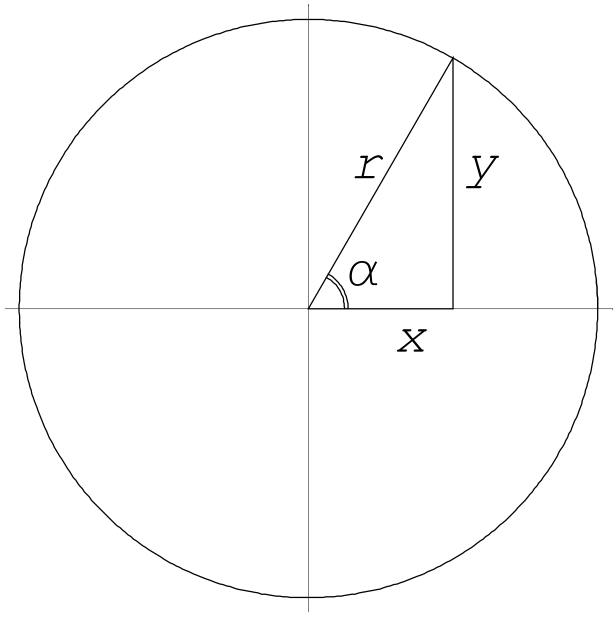
In a reaction to the May version of this paper, a reader objected to the use of the UMA, categorizing it as part of “realistic education in mathematics” as advocated by Freudenthal, and arguing that this kind of education is damaging to the development of mathematical skills and abstract thought. Only the abstract ‘no dimension’ interpretation was considered proper. This objection came as a surprise to me, since I had no intention of such. If it is possible to see this paper as belonging to that Freudenthal approach then it is mere coincidence and I actually cannot vouch for that. And curiously, another reader tends to see some value in the Freudenthal approach. The point however is that this paper only wants to clarify what angles really are. The mentioned reader apparently did not see angles as real lengths. This caused me to include this section.

Thus, a basic notion of analytical geometry is that there is a distinction between ‘no dimension’ (number theory) and ‘unspecified unit’ (space). Logically, that mentioned reader could have no objection when UMA is translated as ‘unit of measurement around’. The objection then centers on whether we should specify what the unit of measurement is. The SI unit is the meter. My suggestion is not to hide that the unit of measurement might well be that SI unit. What this paper proposes is that students become capable in translating specific measurements into a bit more abstract mathematical constructs and vice versa. Since there is length involved, it is required that UMA is mentioned. This will help the student to understand what trigonometry is about. Not mentioning UMA, not explaining what an angle is, withholding the evidence, will hinder the development of abstract thought.

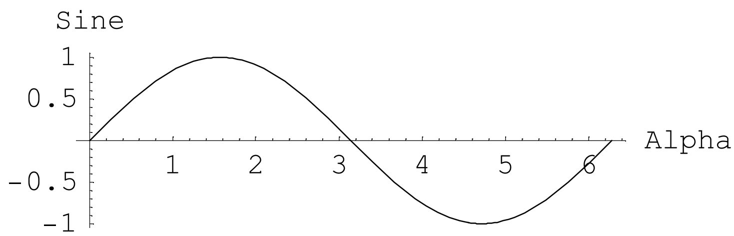
The conventional approach

In mathematical convention, angles are measured counterclockwise, starting at the point $\{1, 0\}$, see diagram 1. The unit circle is defined as the circle with unit radius, i.e. with $r = 1$. The circumference of the unit circle is $\Theta = 2\pi$, with Θ defined as the ratio of the circumference to the radius, or with π defined as the ratio of the circumference to the diagonal, giving the definition of the maximal angle of Θ radians. Conventionally, angle α is written close to the center. We find functions $\text{Sin}[\alpha] = y / r = y$ and $\text{Cos}[\alpha] = x / r = x$. Plotting the sine and cosine function requires zeros at points that are fractions of Θ, see diagrams 2 and 3.

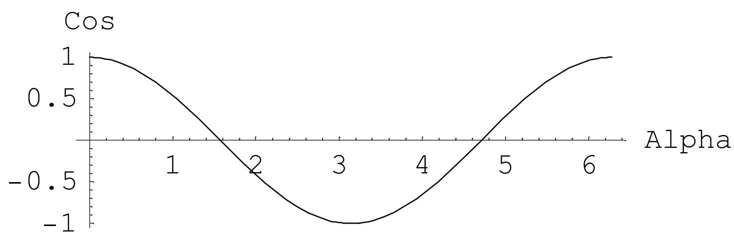
- Diagram 1. The traditional unit circle when $r = 1$, measuring counterclockwise starting from {1, 0}.



- Diagram 2. The sine function for $\alpha = 0$ to $\alpha = \Theta$ radian. Note that the slope is 1 at $x = 0$.



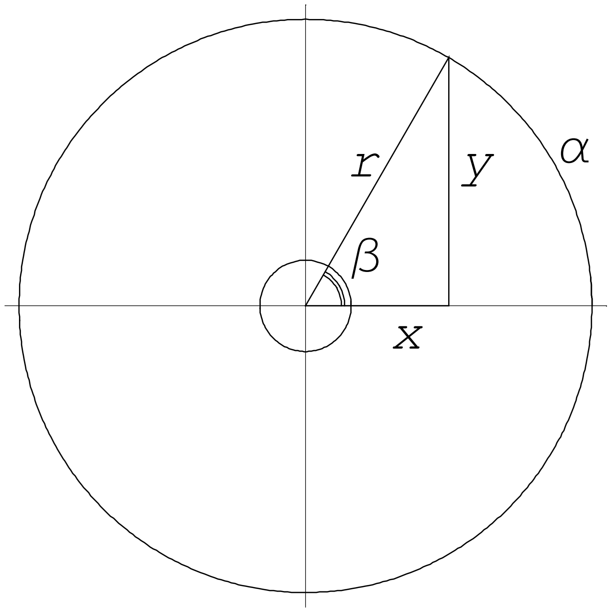
- Diagram 3. The cosine function for $\alpha = 0$ to $\alpha = \Theta$ radian.



The alternative approach

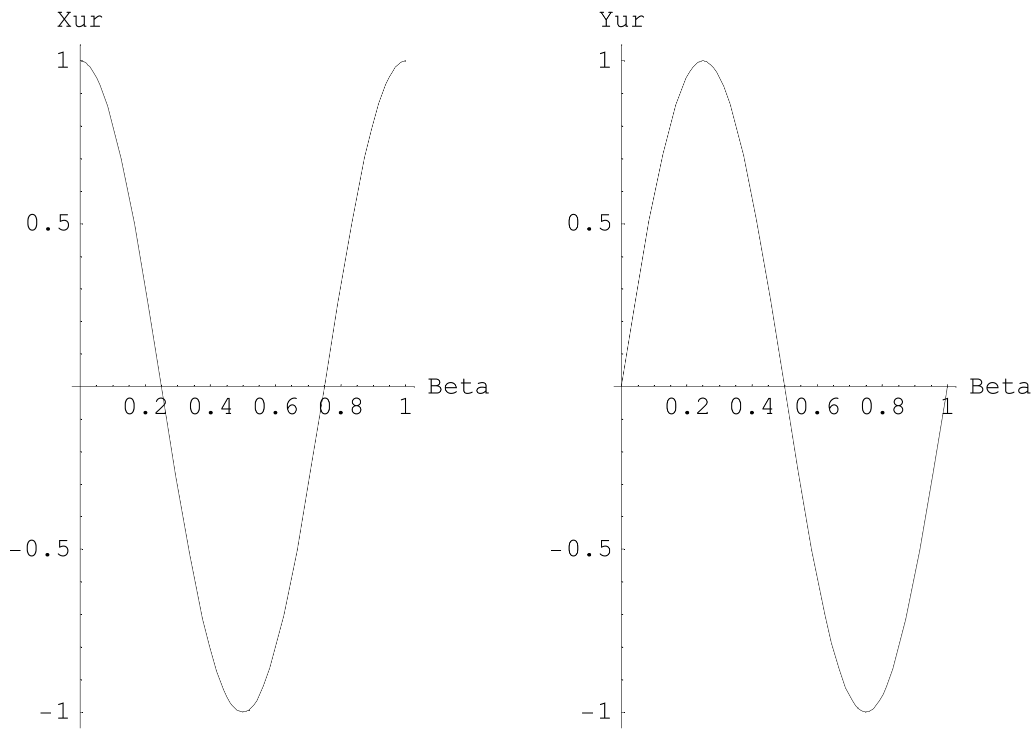
The alternative approach is to define angles on the inner circle with circumference 1, which inner circle has radius 1 / Θ. Whereas α is defined in radians, we now consider $\beta = \alpha / \Theta$ defined in 'unit meter around' (UMA). In this case we write α close to the proper outer arc, and we write β close to the inner arc on the inner circle. PM. An objection to using a circle with circumference 1 can be that one needs a unit length to define π, so that one might feel that one cannot use the former to define the latter. However, when an outer circle has been defined with radius 1 then there must also be an inner circle with radius 1 / Θ.

- Diagram 4. The unit circle when $r = 1$, $\beta = 0$ to $\beta = 1$ UMA.



It will be useful to define the functions x_{ur} and y_{ur} for the co-ordinates of the points on the circle with unit radius (to be distinguished from the circle with unit circumference). Note what this means: while the angle β is defined on the domain of the inner circle with radius 1/Θ and circumference 1, the range values x_{ur} and y_{ur} are on the unit circle with radius 1 and circumference Θ. Thus the functions involve a perceptible transformation and a didactic advantage is the emphasis on the linear way of scaling up. The horizontal co-ordinate is given by the function Xur[β] = Cos[Θ β]. The vertical co-ordinate is given by the function Yur[β] = Sin[Θ β]. See diagram 5.

- Diagram 5. The function Xur[β] = Cos[Θ β] and Yur[β] = Sin[Θ β] for β = 0 to β = 1 UMA.



The notion of the tangent comes rather natural in the discussion of slopes, as Tur[β] = Yur[β] / Xur[β].

Hence:

? Xur

Xur[α] for angle α measured in Unit Meter Around gives the x value on the circle with unit radius. Xur[α] can also be understood as the ratio of the horizontal value to the radius. Since 2 π UMA = 1 radian, we also have Xur[α] = Cos[2 π α]

? Yur

Yur[α] for angle α measured in Unit Meter Around gives the y value on the circle with unit radius. Yur[α] can also be understood as the ratio of the vertical value to the radius. Since 2 π UMA = 1 radian, we also have Yur[α] = Sin[2 π α]

The prime didactic question

The prime didactic question concerns the relation between the sine and cosine of any angle, with whatever direction in the plane, and the sine and cosine of the triangles fixed by the unit circle.

Normally, students are taught the first before the second, e.g. the sine as the ratio of the opposite perpendicular to the hypotenusa. Later, the second is presented, in the context of vectors, polar co-ordinates, the calculation of π, the definition of the radian, the periodic wave functions and the derivatives. The drawback of this order is that students associate sine and cosine with any orientation, so that they have conceptual problems to see these as co-ordinates in a fixed system. The functions x_{ur} and y_{ur} are defined precisely to bridge this gap in understanding.

One might consider to switch the order around. In that case we first present the circles with x_{ur} and y_{ur} and then secondly translate any angle of whatever direction towards the unit circle. The recognition of the perpendicular and the hypotenusa is only required for this step of translation and then measurement sets in by rescaling and using x_{ur} and y_{ur}. This approach seems to have three related drawbacks: (i) the ratios are not recognized on the spot but only after translation, (ii) the perception of space itself might be hindered by continual reorientation, (iii) it may now be that the co-ordinates x_{ur} and y_{ur} are more easily recognized but students may have more problems now to link up to the ratios of any angle of arbitrary direction.

These issues are difficult to decide on. What would be best might transpire in a randomized controlled trial, perhaps running over several generations. We would also need to define what is ‘best’ and how to balance the spatial sense with the handling of the co-ordinates. However, it is the impression of this author that the spatial sense is basic, so that the tradional order of presentation is best, though extended in the first step with UMA angles and in the second step with x_{ur} and y_{ur}.

A more transparant trigonometry

In general

In a course on trigonometry, it remains desirable that students retain a sense of history. It would be insensible to abolish all history and redesign math such that we only retain a pure framework. A sense of history not only allows one to read historical novels but also conveys the sense of wonder, both for the subject itself and the achievements of Babylonian astronomers and Greek axiomatics. A historical sense is also required to be able to communicate with others who haven't heard of Xur and Yur and the UMA metric.

Thus a course outline would be:

1. Historical context. A year has about 360 days, the Sun moves in a circle around Earth, so a circle is divided in 360 degrees. Geometric proof that a triangle has 180 degrees. Thales theorem. Pythagoras. Definition of Sin, Cos and Tan using degrees, for angles without any particular orientation. Explanation of the difference between the definition of an angle and the mentioned two ways of measuring an angle. Stop here. Do not extend with the various trigonometric rules since it is more important to clarify the issue of measurement before such rules become meaningful. Explain that the circumference of a circle can also be measured in UMA, and then clarify that Sin[α Degree] = Sin[α Degree / 360 UMA]. Note that it is important to mention the dimensions since numbers without labels might be easily be confused.
2. The unit circle with Xur and Yur using UMAs. The angles now are oriented, and they are ordered from 0 to 1 UMA. The definition of the circle is Xur[β]^2 + Yur[β]^2 = 1 and similar for any scaled form, so that Pythagoras not only proved his theorem but also provided a definition of the circle (which one can accept without proof). Definition of Θ as the circumference of the circle with unit radius. Define $\pi = \Theta/2$ for the sum of angles in a triangle. With arbitrary radius *r,* the circumference = $\Theta\ r$ and surface = $\Theta\ r^2/\ 2$. The extension of Sin and Cos with radians. Explanation that any angle with whatever orientation can be measured in

UMA or radian or degree, and that these are only scalar multiples of each other. Thus, that Sin[α Degree] = Sin[α rad] = Sin[α] = Sin[α / Θ UMA] = Xur[α / Θ].

3. Clarification to students of the prime didactic question posed in the former section. Our problem in teaching originally is their problem in learning, and it is useful for them to be aware of it.

4. The periodic wave function that arises from plotting Xur or Yur when a point traverses the circle counterclockwise, taking the arc (angle) as the explanatory variable. Transformations of that function, with starting point, equilibrium, period and amplitude. Conclusion that only one function is necessary since the other can be translated into it, and hence use only one, preferably Xur since it is on the horizontal axis. (Note that some authors write Cos[x] but then they forget that they should write Cos[α], since they already used x = Cos[α]. There seems to be a convention that any function can have an x as the variable for this domain, but logically this cannot be maintained when there are functions of functions.)

5. The various trigonometric rules, geometrically proven in terms of Xur and Yur. E.g. the 'xur rule' (instead of the 'cosine rule') and the 'yur rule' (instead of the 'sine rule'). The reason to use Xur and Yur is that they undisputedly use UMA and thus are less cluttered with π.

6. Some geometric problems that clarify that the different methods of measuring angles can complement each other in finding solutions for lengths, surfaces and contents.

7. The handling of co-ordinates as vectors. Addition and multiplication of co-ordinates. Polar co-ordinates and complex plane. Euler's equation. Multiplication by i is a quarter turn along the circle.

8. The derivatives. We find that $\frac{d}{d\alpha}$Xur[α] = - Θ Yur[α] and $\frac{d}{d\alpha}$Yur[α] = Θ Xur[α] because of the scale factor. Note that both UMA and radian use the true length (e.g. the meter) as the unit of measurement, so that in the derivative the arc approaches the chord, but the difference between UMA and radians resides in the size of the radius in the denominator. Thus there is a similar effect as with the derivatives of e^x and a^x, where the first remains the same while the second gets a ln[a] coefficient. In practice, as soon as the variable becomes a somewhat more involved expression then the additional coefficient does not matter much in added complexity. Nevertheless, where we first saw a reduction in mention of Θ this now appears to be a place where it will show itself more often. This property implies that radians and Sin and Cos defined on the radians will remain in use, especially for heavy users of calculus. To students it must be explained that it would be

feasible to first present Xur and Yur and then present Cos[α] = Xur[α / Θ] and Sin[α] = Yur[α / Θ] as the scaled versions with the sometimes more attractive derivatives - but that this order of presentation has not been chosen because of the spatial sense, referred to in point 3.

A first course would contain these aspects and give a general overview, so that it should become clear how these aspects are linked. Subsequently, one can imagine follow-up courses that handle the various aspects in more depth.

(PM. There is another didactic link between sine and the exponential function. When it is explained that the sine is just the y-value, then students tend to react with "oh, it is just the y-value". When it is explained that the logarithm is just the exponent, then students tend to react with "oh, it is just the exponent". For this reason it is advisable to replace the opaque term 'logarithm' (log) with 'recovered exponent' (rex). When taking the power $y = \text{base}^x$ then the exponent disappears, but it is recovered with x = rex[y, base]. PM. Another link is that some authors write $\sin^{-1}[\alpha]$ for the inverse and $\sin^2[\alpha]$ for the squared value, in the same way as they do for logs, while they don't seem to realize that if $\sin^2[\alpha] = \sin[\alpha]^2$ then $\sin^{-1}[\alpha] = \sin[\alpha]^{-1}$ which is not the ArcSin. In cases like this, it is better to stick to the notion that f^{-1} is the inverse if f is the name of a function, so that one must use $\sin[\alpha]^2$. Likely it is advisable to use both ArcSin and $\sin^{-1}$ since the first expresses more clearly that the ratio is translated back to an arc while the second links to the general notion of an inverse.)

The following examples indicate that trigonometry has become more transparent.

Example 1: Standard angles

Historical values are the angles all 1/12th apart (with the 12 months in a year (e.g. of 360 days) or the hours on the dial) or at 1/8th apart (corners of the wind). We can just use the fractions and we don't have to multiply these by 360 or 2π.

```
angles = Union[ Range[0, 6] / 12, Range[0, 4] / 8];
(# /@ angles & /@ {Identity, Xur, Yur, Tur}) /. ComplexInfinity → ∞
```

$$\begin{pmatrix} 0 & \frac{1}{12} & \frac{1}{8} & \frac{1}{6} & \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{5}{12} & \frac{1}{2} \\ 1 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2} & -1 \\ 0 & \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} & 1 & \frac{\sqrt{3}}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 1 & \sqrt{3} & \infty & -\sqrt{3} & -1 & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}$$

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Solve[Xur[α] == -Sqrt[3]/2, α]
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$$\left\{\left\{\alpha \to -\frac{5}{12}\right\}, \left\{\alpha \to \frac{5}{12}\right\}\right\}$$

It will be informative to also consider the decimal spots.

```
angles = Range[0, 5] / 10;

(# /@ angles & /@ {Identity, Xur, Yur, Tur}) /. ComplexInfinity → ∞
```

$$\begin{pmatrix} 0 & \frac{1}{10} & \frac{1}{5} & \frac{3}{10} & \frac{2}{5} & \frac{1}{2} \\ 1 & \frac{1}{4}\left(1+\sqrt{5}\right) & \frac{1}{4}\left(-1+\sqrt{5}\right) & \frac{1}{4}\left(1-\sqrt{5}\right) & \frac{1}{4}\left(-1-\sqrt{5}\right) & -1 \\ 0 & \frac{1}{2}\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)} & \frac{1}{2}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)} & \frac{1}{2}\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)} & \frac{1}{2}\sqrt{\frac{1}{2}\left(5-\sqrt{5}\right)} & 0 \\ 0 & \frac{\sqrt{2\left(5-\sqrt{5}\right)}}{1+\sqrt{5}} & \frac{\sqrt{2\left(5+\sqrt{5}\right)}}{-1+\sqrt{5}} & -\frac{\sqrt{2\left(5+\sqrt{5}\right)}}{-1+\sqrt{5}} & -\frac{\sqrt{2\left(5-\sqrt{5}\right)}}{1+\sqrt{5}} & 0 \end{pmatrix}$$

Example 2: Trigonometric rules

Pythagoras gives:

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Xur[β]^2 + Yur[β]^2 // Simplify
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1

Consider the sine rule for the triangle *ABC* with angles α, β, γ and sides *a, b, c.* Drop a perpendicular from *C* to *c*, with height *h.* The sine rule uses sin(α) = *h / b* and sin(β) = *h / a* and now we get:

```
{Yur[α] == h / b, Yur[β] == h / a}
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and this gives Yur[α] / *a* = Yur[β] / *b* (now in UMA instead of radian).

Example 3: Solving equations

A typical question is: Solve $\cos(x)^2 - \cos(x) = 0$. Solved by $\cos(x)\,(1 - \cos(x)) = 0$. $\text{Cos}(x) = 0$ or $\cos(x) = 1$. Thus $x = \pi/2 + k\,\pi$ or $x = 2k\pi$.

This now becomes: Solve $\text{xur}(\beta)^2 - \text{xur}(\beta) = 0$. Solved by $\text{xur}(\beta) = 0$ or $\text{xur}(\beta) = 1$. Thus $\beta = 1/4 + k/2$ or $\beta = k$ (UMA).

Note that while π clutters the traditional expression, that traditional expression also implicitly uses π to indicate the kind of variable. The traditional approach does not explicitly state that its solution is in radians while it is neater to actually state that. (Also note the use of x while it should have been α.)

Example 4: Calculating arcs

A typical question is: Consider points P = {x, 0.8} and Q = {0.1, y} on the unit circle for negative solutions of x and y. Calculate the length of the shortest arc from P to Q. Calculated by: ArcSin[0.8] = 0.927. The angle to P is $\alpha = \pi$ - 0.927 = 2.214. ArcCos[0.1] = 1.471. The angle to Q is $\beta = 2\pi$ - 1.471 = 4.813. The shortest arc between P and Q is β - α = 2.60.

This now becomes: Done by υ = ArcYur[0.8] = 0.147584. The angle to P is α = 1/2 - υ = 0.352416. ξ = ArcXur[0.1] = 0.234058. The angle to Q is β = 1 - ξ = 0.765942. The shortest arc on the unit circle is Θ (β - α) = 2.60.

Example 5: Derivatives

A typical question is: Consider the function $f[\alpha]$ = 2 cos[α] + sin[2α] + 1 in the domain [0, 2π]. Calculate the global minimum and maximum.

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Plot[2 Cos[α] + Sin[2 α] + 1, {α, 0, 2 π}, AxesLabel → {α, f[α]}];
```
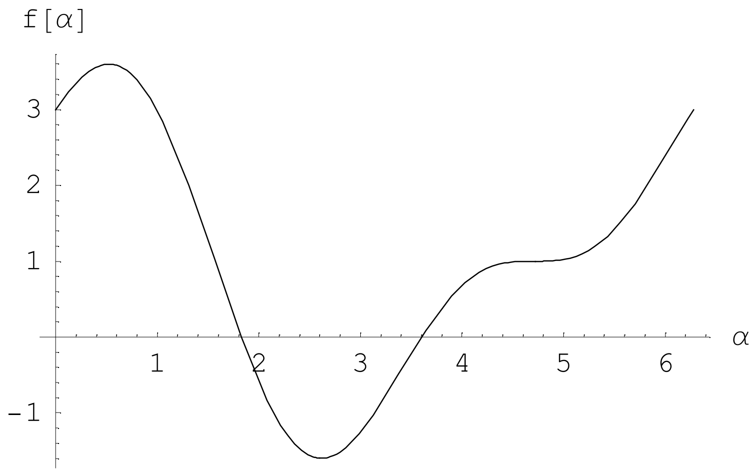



This is answered by:

$f'[\alpha]$ = -2 sin[α] + 2 cos[2α] = 0

sin[α] = cos[2α]

sin[α] = sin[π/2 - 2α]

$\alpha = (\pi/2 - 2\alpha) + 2\pi k \quad \lor \quad \pi - \alpha = (\pi/2 - 2\alpha) + 2\pi k$

$3\alpha = \pi/2 + 2\pi k \quad \lor \quad \alpha = - \pi/2 + 2\pi k$

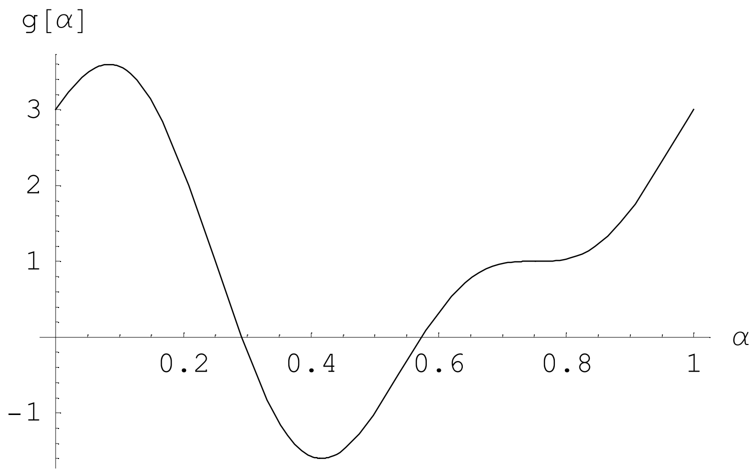
$\alpha = \pi/6 + 2/3\, \pi\, k \quad \lor \quad \alpha = - \pi/2 + 2\pi k$

Requiring $0 \leq \alpha \leq 2\pi$ gives $\alpha \in \{\pi/6, 5/6\, \pi, 3/2\, \pi\}$.

With reference to the graph, the global maximum is at $\alpha = \pi/6$ and the global minimum at $\alpha = 5/6\, \pi$. QED

A similar question can be formulated in terms of Xur and Yur. Consider the function $g[\alpha] = 2\ x_{\text{ur}}[\alpha] + y_{\text{ur}}[2\alpha] + 1$ in the domain [0, 1]. Calculate the global minimum and maximum.

```
Plot[2 Xur[α] + Yur[2 α] + 1, {α, 0, 1}, AxesLabel → {α, g[α]}];
```



This is answered by:

$g'[\alpha] = -2\ \Theta y_{\text{ur}}[\alpha] + 2\ \Theta\ x_{\text{ur}}[2\alpha] = 0$

$y_{\text{ur}}[\alpha] = x_{\text{ur}}[2\alpha]$

$y_{\text{ur}}[\alpha] = y_{\text{ur}}[1/4 - 2\alpha]$

$\alpha = (1/4 - 2\alpha) + k \quad \lor \quad 1/2 - \alpha = (1/4 - 2\alpha) + k$

$3\alpha = 1/4 + k \quad \lor \quad \alpha = - 1/4 + k$

$\alpha = 1/12 + k / 3 \quad \lor \quad \alpha = 3/4 + k$

Requiring $0 \leq \alpha \leq 1$ gives $\alpha \in \{1/12, 5/12, 3/4\}$.

With reference to the graph, the global maximum is at $\alpha = 1/12$ and the global minimum at $\alpha = 5/12$. QED

Example 6: Euler's equation

In presenting the Euler equation, it is helpful to start with z = Cos[α] + i Sin[α] so that $\partial z / \partial \alpha$ = - Sin[α] + i Cos[α] = i z. Since we know that e^x has itself as the derivative, and this apparently also happens with complex numbers though with coefficient i, we can express the one into the other. In this case, sine and cosine may remain superior expressions because of the derivatives, but then this will be appreciated for this very reason (and it will not be something not noted).

Euler's equation for the complex plane can be directly translated to Xur and Yur.

E^(I α) == Cos[α] + I Sin[α] == Xur[β] + I Yur[β] /. $\alpha \to \Theta\, \beta$

$$e^{i \beta \Theta} = \cos(\beta \Theta) + i \sin(\beta \Theta) = \cos(2 \pi \beta) + i \sin(2 \pi \beta)$$

We usefully stick to i = {0, 1} so that the transformations are done along the unit circle while the angle is measured on the UMA circle. Hence we get $e^{i \Theta \beta}$ = Xur[β] + i Yur[β].

PM. A point to remember is that $e^{i\pi}$ = -1 represents an operator, so that it is not correct to deduce

E^(I α) == E^(I 2 π β) = ((E^(I π))^2)^β == ((−1)^2)^β == 1^β = 1

Another way to understand this is to consider the dimensions and see that we cannot 'split off' β:

E^(I α Radian) == E^(I (2 π Radian / UMA) β UMA)

Euler's relation $e^{i\pi}$ = -1 is defined for radians and not for radian / UMA. The replacement by -1 in above 'deduction' is not valid.

Conclusion

The conventions in trigonometry once seemed innovative at their time of introduction but when we reconsider the subject then some more improvements appear to be possible. Didactic issues in trigonometry are the opaque names of sine and cosine and the cluttering of questions with π or 360 whereas a simple 1 suffices. The solution is to use the 'unit meter around' as the yardstick for angles and to use the x_{ur} (Xur) and y_{ur} (Yur) functions for the $\{x, y\}$ co-ordinates on the circle with unit radius. The relevant mathematical constant is $\Theta = 2\pi$ rather than π and it comes into use much less when we use UMAs instead of radians. These issues can be resolved together at the same time and they can indeed be resolved with relatively little effort.

Current trigonometry is needlessly torturing our students. The torture derives mainly from conventional thinking and not from the math itself. So students arguing for a more transparent trigonometry have math on their side.

References

Colignatus is the name of Thomas Cool in science.

I thank Robert Palais, Peter Kop and Abraham Roth for comments on an earlier version. All errors remain mine.

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