Proportions and fractions (as used in "Elegance with Substance")

Thomas Colignatus, July 12 2009

Abstract

Proportions and fractions are used in my book "Elegance with Substance" but not explicitly defined. This paper gives explicit definitions and clarification, and includes the topic as another case in support of the main conclusion of the book.

Introduction

H. Pot (2009abcd) calls attention to the confusion in education on proportions and fractions. He lists examples in teaching material – even for the education of teachers in primary schools – where it is suggested that proportions and fractions would be the same, while they are not.

In the Dutch TAL project, see KNAW (2003) and Freudenthal Instituut (2009), it is stated for an audience of students who want to become primary school teachers: "Fractions, percentages, decimal numbers, and proportions are different descriptions of something that we can regard *in some respect* as the same." (my translation from Pot (2009d) and my italics). The italized qualifier makes this alright (though vague) but the surprise is that Pot reports that the authors did not want to go into detail "because of the targetted readership". This is surprising since we definitely would want primary school teachers to understand the distinction between proportion and fraction.

This situation is an example that might be included in (a second edition of) "Elegance with Substance" (EwS) (Colignatus 2009) as yet another case where the education in mathematics is not as we would wish it to be, and another argument why a parliamentary inquiry would be useful.

On content, EwS does not explicitly deal with the distinction between proportions and fractions. EwS highlights (1) the problem of 2 + ½ versus 2 ½ (EwS:20), and (2) the problem of the static quotient y / x and the dynamic quotient $y // x$ (EwS:27). However, by implication, EwS refers to proportions and fractions. It will be useful to discuss here what the distinction is (as it is used in EwS). Clarity on y : x, y / x and y // x. contributes to a better understanding of calculus and the differential quotient.

This present short paper thus supplements EwS on the issue of proportion and fraction.

History

Fractions are important. The Egyptians had only fractions ½, 1/3, 2/3 and ¾, and their mathematics stagnated for 3000 years on this.

The distinction between proportions and fractions is influenced heavily by history. Pythagoras assumed that all phenomena could be measured in ratio's of natural numbers and was horrified when one of his students showed that this was impossible for √2 (EwS:34). Henceforth, the Greeks solved their problem by switching to geometry where such quantities can be dealt with easily. By consequence, the Greeks developed a theory of proportions and did not develop the theory of arithmetic (and subsequently algebra). Another consequence is that this theory of proportions entered the textbooks on mathematics, likely to stay there forever, even though there is now a better theory of arithmetic.

To understand the distinction between proportions and fractions we thus need this historical overview, where the theory of proportions is a more primitive or simple theory while arithmetic is more powerful and develops (a) number theory and (b) division (giving fractions) as the inverse of multiplication. When these two theories (frames of reference) are not distinguished then confusion enfolds.

It is a moot question whether it is wise to teach these two theories and their history in primary school. But proportions are endemic in our language and culture, and relevant when we scale items up or down, and thus it seems that the issue cannot be avoided.

Proportion space

EwS:21 contains a graph of "proportion space", reproduced below.

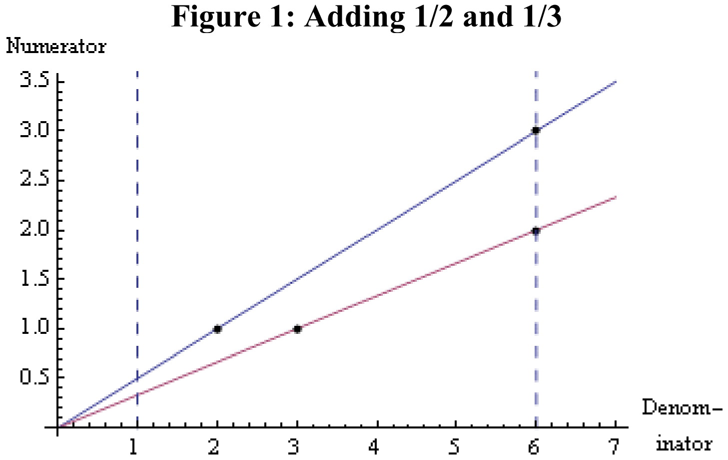
A *proportion* (ratio) is a point in proportion space. A proportion 3 : 5 associates with the point {5, 3} in proportion space. The *fraction* 3 / 5 is the slope of the line through {5, 3} and the origin. These two concepts now are well-defined. Proportion is two-dimensional while fraction is a *number* on the real number line.

(In itself this distinction might not quite hold. Babylonian numbers might be seen as multidimensional phenomena as well … But in standard theory we start with natural numbers, and then include zero, negatives, fractions (rational numbers) and irrationals to create the number line.)

There is a problem with the concept of "same". Since the point {5, 3} is not the same point as {10, 6} we may wonder whether 3 : 5 and 6 : 10 are the "same" proportions. For fractions it is standard that 3 / 5 = 6 / 10 are identically the same number, but for the two-dimensional ratio's we need the expression 3 : 5 ~ 6 : 10 to show equality other than pure identity. For this reason it is better to say that 3 : 5 *associates* with {5, 3} in proportion space rather than that it is identically this point.

Thus, the primitive / simple theory of proportions (frame of reference) comes with a notion of equality that differs from identity. In the more advance theory of arithmetic this is replaced by a notion of equality that is identity. (Two line sections of the same length need not be identically the same line sections but their lenghts would be identically the same.)

Figure 1: Adding 1/2 and 1/3



Dutch idiosyncracy

Here arises an idiosyncracy of the Dutch language. Simon Stevin (1548-1620) enriched Dutch language with specific Dutch translations of Latin and Greek terms – see EwS:17 and see the Appendix below. Other European languages still have "parallel lines" as in Greek but Dutch has "evenwijdige lijnen" where Stevin thought that "lijn" was sufficiently infused into Dutch but replaced "parallel" with "even-wijdig", where "wijdig" means English "width-y". He also replaced proportion (ratio) with "reden", so that the "same proportions" translate into Dutch as "even-redig". This seems to have the advantage in Dutch that the proportions 3 : 5 and 6 : 10 are not the "same" in the sense of "identical", but are merely "evenredig" – "equally proportional". Thus in Dutch there need not be a confusion between identity and equality other than identity. However, the twist is that Davidse (2009) explains that Stevin meant that "even" stood for the word "equal". Also the English "even" and "uneven" would relate to the Germanic root for equality. An "even number" namely can be split into two equal parts. Hence, Stevin's invention of "evenredig" only increases the confusion in Dutch since it obfuscates the notion of "same proportion". The Dutch may think that same proportions are not the same but merely "evenredig" and then they lose out, in this subject area, on clarity between "equality" and "identity". Also, surprisingly, the modern Dutch word for algebraic identity ("is gelijk") ($a = b$) was coined by Stevin from the Latin "similes" (English "similar") which differs from Latin "aequales" (English "equal"). Thus there has been a shift of Dutch meaning over the ages as well, with perhaps confusing colloquial Dutch expressions that infuse the classroom.

Educational terms

These (theory-dependent) definitions for proportion, fraction, identity and equality (other than identity) are pure theoretical developments and suffice for theory. Next there is education, and education may require additional terms for the communication between teacher and student.

For example, some may wonder whether 2 + ½ is a fraction, and then the term "fractional number" might be used. This also relates to the transformation from the theory of proportions to the theory of arithmetic where a fraction is interpreted as a *number* on the real axis (and instead of "rational number" it is called "fractional number").

For example, the *form* y / x might be called "fraction" and the *outcome* might be called "fractional number". In that case the textbook might say: the outcome of a fraction is a fractional number.

An alternative would be to keep "fraction" for the number, and use "fractional form" for the form.

But actually, 1 / 2 is *both* a form *and* the correct denotation of the number. Thus the educational terms must be used so that they do not create new confusions.

Some pupils or students will regard the decimal expansion 0,5 as the number so that the fraction 1 / 2 is only a form, or instruction for calculation. (PM. See EwS:34.) Indeed, educational simplicity arises when this approach is adopted. At that level of educational simplicity we might indeed forget about the distinction between proportions and fractions, and only use fraction or proportion 1 / 2 and number 0,5. We then only have numerical equality and forget about identity. The only problem might be that textbooks in primary education use different definitions than textbooks in secondary education and up. One supposes that "evidence based education" can clarify what kids can handle. However, such research needs to be subtle. Pierre van Hiele has emphasized that primary schools can handle vectors. Thus we need research that does not obfuscate what is possible. Note by the way that above "proportion space" diagram is a bit different from a normal vector space and thus it is necessary to introduce it at exactly the same time with a vector diagram, to prevent future confusion.

Algebra

For algebra, it is useful to manipulate *y / x* or *y // x* as a form. If it is thought of as a number only then it might be hard to see how it can be manipulated. For example $(2\ \pi) / \pi$ is a fraction of two irrationals and thus would seem to be irrational too, but algebra shows that it is rational.

Thus it remains useful to distinguish both form and number. But only as angles to consider the same expression.

Verb versus noun

EwS:64 has: "Apparently fractions in Chinese are clearer too. Instead of *two-fifths* it would use *two-out of-five*. First creating *fifths* indeed is an additional operation. Perhaps the West is too prim on the distinction between the ratio 2:5 and the number 2/5. Perhaps it does really not make a difference except in terms of pure theory - the verb of considering the ratio and the noun of the result (called "number" when primly formalized in a number theory)."

Thus (1) "ratio" is a verb in the theory of proportions, and (2) "dynamic quotient" is a verb in the theory of arithmetic.

Perhaps there is didactic advantage in translating or projecting the historical distinction between the theory of proportions and the theory of arithmetic into a distinction into verb and noun. This is not a question that we can resolve just here.

Some related points

With the burden of history, we cannot avoid tradition and convention. EwS:23 has: "The mathematical symbol π (Greek "pi") is defined on a circle as the ratio of the circumference to the diameter. Angles are measured in 360 degrees or 2 p radians." This follows the convention of using the ratio instead of concentrating on the number on the number line. This is no problem once it is obvious that a ratio can be projected into a number. But it may be confusing when that is not understood.

EwS:35 discusses the slope of a line. When the distinction between proportion and fraction is not clear then the notion of a slope will be confusing too.

There is a useful distinction between a proportion or ratio between variables of the same dimension and a rate of one dimension versus another. The distinction is not essential for the present discussion.

Additional answer to the problem highlighted by Pot

The above basically provides an answer to the problem as it has been highlighted by Pot (2009abcd). Proportions and fractions are not the same, but at some phase in education perhaps they might be taken as the same in order not to overburden the pupils.

In addition, Pot suggests to take proportionality as a basic concept, primitively understood by the human mind, and to subsequently develop the notion of number from there. This is converse to the standard approach to take number as basic and subsequently develop proportionality. However, when the confusion has been resolved by better understanding the historical development of the rational and irrational numbers, and the distinction between the theory of proportions and the theory of arithmetic, then there is no need to change the standard approach.

Also, Pot (2009c) has provided many useful quotes also from old textbooks some dating ages ago. Who is interested in this subject will enjoy to see the struggle with the confusion. Though it may also be that the textbook writer has deliberately chosen ‘for educational purposes’ to take proportions and fractions the same.

Conclusion

Proportion and fraction are well defined, and have their related but own theories. A highschool graduate and by implication a primary school teacher should be able to understand and reproduce these theories and definitions. It remains a choice what is taught at what phase in education. Given good didactics and following Pierre van Hiele, it should be possible for a large section of the population in primary school to learn both vectors and the distinction between proportions and fractions.

Appendix on Simon Stevin and Dutch words

Form Davidse (2009):

> Aequales - Even (voor ons dus: gelijke, Engels: 'equal')
> Similes - Ghelijcke (wij denken dan: gelijkende, zoals in: dergelijke)
> Parallela - Evewydeghe
> Proportio - Everedenheyt
> Ratio - Reden (verhouding, bv: van de zwaarste tot de lichtste, Engels: 'ratio')

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