Comparing two approaches to calculus: "direct" and algebra

Comparing the approaches by (1) Shen & Lin (2014) "Direct Calculus", and (2) Colignatus in "A Logic of Exceptions" 2007 and "Conquest of the Plane" 2011

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Abstract

Shen & Lin (2014) (reporting over two decades) and Colignatus (2011) propose to develop derivative and integral in simultaneous manner, thus no separate chapters but both in one chapter by reversible steps, so that at the end the Fundamental Theorem of Calculus is an obvious achievement. One has to start somewhere though. Shen & Lin (2014) start with the trigonometric tangent or derivative and take the height increment as the primitive. Colignatus (2011) starts with surface and deduces the derivative on area change that appears to have the form of the trigonometric tangent. The expositions have other aspects but these should not distract from the main insight that a simultaneous development comes with great advantages. Randomised controlled experiments should show what approach works best for students.

1. Introduction

The following comparison will look at these angles on calculus:

- didactics
- content, with the possibility of a fundamental redesign of calculus.

Ancient Greek geometry is called *synthetic* since one generates proofs by "putting together" the various givens (definitions, axioms and earlier theorems). In *analytic geometry* one provides proofs by decomposing (analysing) issues in terms of algebra. A subsequent historical development was that even analytic geometry and its system of co-ordinates was seen as not exact enough, whence one looked for foundations in arithmetic. This became the field of *analysis.* Corner stones of the latter are notions of numerical continuity and limits. The current perception in mathematics is that calculus can only be done in *analysis*.

Notions of continuity and limit make analysis and calculus (didactically) complicated. An approach is to only teach the rules, but this sacrifices both rigour and understanding what a derivative and integral actual are. Thus the stage is set for a major redesign. This redesign would first of all be in terms of didactics but might also be of fundamental nature, as an answer to the switch to analysis.

Elsewhere I compare with the approach by Michael Range, see Colignatus (2017e). The effort by Alain Schremmer to revive Lagrange's approach must also be mentioned. [1]

Here I will look at the approach by Shen & Lin (2014). They label their method as "direct calculus". I wonder whether they would be willing to revise this label, since their implication would be that other methods would be "indirect", and this might cause a needless discussion on criteria for "directness". A better expression might be the "simultaneous development of integral and derivative". PM. The term "direct integral" is in use for other applications. [2]

The definition of the dynamic quotient can be found in Colignatus (2007:241) or (2011:57). It appears that a mere reference is not sufficient and that some readers require that a discussion is self-contained. This condition is awkward since COTP really deserves a study because of its approach to didactics and essential refoundation of calculus. Yet, a section below thus repeats the definitions of the dynamic quotient and the derivative that uses this.

2. Overview of the comparison

Table 1 gives an overview of the comparison in this paper.

Table 1. Overview of the comparison

	Shen & Lin (2014)	Colignatus (2007+)
Scope	Two hours introduction to calculus for non-math-majors	Didactic development of analytic geometry and calculus for highschool and matricola for non-math-majors, primer for teachers, essential redesign of calculus
Limits	Required fundamentally, and also didactically onwards from exponential functions and trigonometry	Not required for polynomials, exponential functions and trigonometry. Currently required for more. Obviously required for math majors.

[1] http://www.freemathtexts.org

[2] https://en.wikipedia.org/wiki/Direct_integral

Simultaneous introduction of both derivative and integral	Key notion	Key notion
One has to start somewhere	Slope of a road	Surface under constant function and line
Diagram of with slope	Yes, but without secant and area, reference to height increment	Essential reliance on surface while formulas show the simultaneous relevance of derivative and integral
Notion of tangency	Reference to the double root line. Conflicting statement on slope of line vs slope of curve	Standard definition of tangent as the line that adopts the slope of the curve
Touching or overlapping with the curve ?	Mention that "tangent" comes from Latin "touching"	Proposal for better name "incline" since the tangent may also cut the curve
Use of double root	Factorisation is tedious (p13)	Not aware of the method at until December 2016. Now, might mention it, but not spend much time on this (simplification only required for one step, then move on to the standard rules)
Handling of area	They define "area" as what is measured by the integral, but this cannot be done since the notion of area is already defined by the square unit	The trick is to assume that *f* gives a surface under some function *g,* and that it is the objective to find this *g.* Thus surface is already given, and we are only interested in the relation between integral and derivative, making sure that each step is reversible.
Main criticism	(1) There is no proof that the double root line generates the slope of the curve (2) Why use double root as introduction to calculus, when the focus is on derivative (slope) and integral (increment, surface) ? (3) The two hour course still contains the Mean Value Theorem and uses this to define the derivative in limit form. If this is feasible, why not do so from the start ? (4) In what manner is the criticism in the introduction w.r.t. the standard approach answered exactly ? What steps find students crucial improvements for their understanding ?	There may be some criticism but it basically turns out then that there is bad reading
Minor points	(1) "Antiderivative" is used for "indefinite integral" and "integral" is used for "definite integral". I	See Colignatus (2011b)

	would prefer traditional terms, notable "primitive". [3] Apparently, Wolfram Alpha has served the use of "antiderivative" but, still, use this wisely. [4] (2) "Direct calculus" suggests that other methods are "indirect". It is better to find another label, that is informative to students too, like "simulateous developement".	

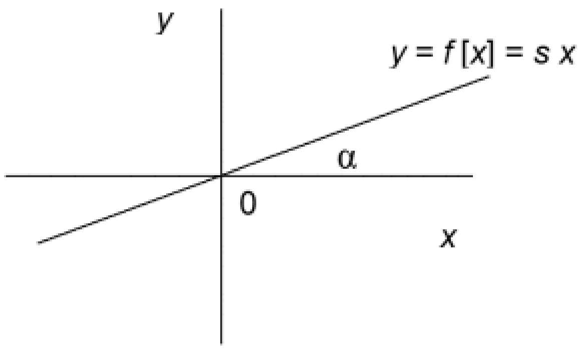
3. *Short restatement of dynamic quotient and derivative*

The following basically repeats sections from Colignatus (2016ad) (2017de). See COTP for the theoretical development and the approach to calculus in general (integral and derivative).

3.1. Ray through the origin and definition of dynamic quotient

Let us consider a ray – rays are always through the origin – with horizontal axis x and vertical axis y. The ray makes an angle α with the horizontal axis. The ray can be represented by a function as $y = f[x] = s\ x$, with the slope $s = \tan[\alpha]$. Observe that there is no constant term ($c = 0$). See Figure 1.

Figure 1. A ray with angle α and slope *s*



The quotient y / x is defined everywhere, with the outcome s, except at the point $x = 0$, where we get an expression 0 / 0. This is quite curious. We tend to regard y / x as the slope (there is no constant term), and at $x = 0$ the line has that slope too, but we seem unable to say so.

There are at least five responses:

(i) The argument can be that y has been defined as $y = s\ x$, so that we can always refer to this definition if we want to know the slope of the ray. This approach relies on a notion of a "memory of definitions", to be used when algebra lacks richness in expressiveness.

(ii) Standard mathematics can take off with limits and continuity.

(iii) A quick fix might be to redefine the function with a branching point:

[3] http://math.stackexchange.com/questions/47703/how-common-is-the-use-of-the-term-primitive-to-mean-antiderivative
[4] http://mathworld.wolfram.com/Antiderivative.html refers to "indefinite integral" too.

$$\text{SlopeOfRay}[y, x] = \begin{cases} y / x = s & \text{if } x \neq 0 \\ s & \text{if } x = 0 \end{cases}$$

We can wonder whether this is all nice and proper, since we can only state the value s at 0 when we have solved the value elsewhere (or rely on the definition as in (i) again). If we substitute y when it isn't a ray, or example $y = x^2$, then we get a curious construction, and thus the definition isn't quite complete, since there ought to be a test on being a ray. Anyway, defining lines in this manner isn't a neat manner. It is really so, that we cannot define a line as $y = s x + c$ and that we must specify the branching when $x = 0$?

(iv) The slope y / x is regarded as a special case of "rational functions". See the section above and the discussion of Range (2016c:16) in Colignatus (2017e). If we work on coefficients only, then we get Ruffini's Rule (a case of "synthetic division"), see Colignatus (2016ef) also referring to MathWorld. [5] The first problem is that in this approach the issues of "identifying the factors" and "adjusting the domain" are only indicated and not made explicit via separate notations. The term "synthetic division" indicates that it might not be "proper division". To what extent is there proper division, so that "eliminating" the factor x generates a result that can be understood as the slope of the line at that point (i.e. fitting to the tangent in trigonometry) ? The second problem is that this remains within the realm of polynomials.

(v) The algebraic approach uses the following definition of the dynamic quotient. Let y / x be as it is used currently in textbooks, and let y // x be the following process or program, called *dynamic division* or *dynamic quotient*, with numerator y and denominator x:

> y // x ≡ { y / x, unless x is a variable and then: assume $x \neq 0$, simplify the expression y / x, declare the result valid also for the domain extension $x = 0$ }

Thus in this case we can use y // x = $s x$ // x = s, and this slope also holds for the value $x = 0$, since this has now been included in the domain too.

We thus *extend the vocabulary of algebra*, so that *multiplication with variables* gets an inverse with *dynamic division by variables*. Since this is a new suggestion we must obviously be careful in its use, but the application to the derivative is a case that appears to work.

The case of the line may be seen as a special case of a polynomial. However, the general notion is "simplify", and there might be other ways than just eliminating factors.

3.2. Dynamic quotient has the denominator as a variable

Simplification only applies when the denominator is a variable but not for numbers. Thus x // x = 1 but 4 // 0 generates 4 / 0 which is undefined. Also x / x is standardly undefined for $x = 0$.

This definition assumes a different handling of different parts of the domain. The test on the denominator is a syntactic test. When the denominator is an expression like (p + 2) then the syntactic test shows that the denominator is a variable, $x = p + 2$. One does not substitute "(p + 2) is a variable" for substitution doesn't look at syntax but uses the value of the variable.

It has been an option in the {...} definition above to write "(a) variable" instead of "a variable", which allows a shift from the syntactic test towards the semantic test of variability, and which also allows *substitution* into the definition, like "(p + 2) is (a) variable". After ample consideration, already in 2007 and later explicitly in Colignatus (2014), I think that we are better served with the syntactic test on the denominator, since this directly leads to the question: what is the domain of the denominator ?

[5] http://mathworld.wolfram.com/RuffinisRule.html

The use of the curly brackets {...} also borrows from *Mathematica.* The brackets signify a list, that can be a set, but when the elements are expressions then the sequential evaluation of those turns into a programme.

3.3. From eliminating factors in polynomials to general "simplification"

In multiplication, (x – 1) (x + 1) = (x^2 – 1) holds for all real x. For division we lack an efficient vocabulary to express (x – 1) = (x^2 – 1) / (x + 1), since this is undefined for x = -1. We can introduce branching, but still would have to use a limit to recover the value at x = -1. When we want to identify or *isolate* the factors however then this "isolate" would commonly be tantamount to requiring division.

An alternative way to identify factors (and find the derivative) for polynomials is the use of coefficients and Ruffini's Rule. If multiplication for polynomials is equivalent to manipulating coefficients, then the latter can also be used for the reverse process of division. See Colignatus (2016ef), that was inspired by (with thanks to) Harremoës (2016) also linking to Bennedsen (2004). It works for polynomials but is it general enough, for non-polynomials ?

There remains the notion of a slope however too. There is no clear link between coefficients (Ruffini's Rule) and the slope. We find the proper *values*, which suggests that there is such a link, yet this link must be shown. The method may be an efficient calculation method, but it doesn't explain that when we find outcome *s,* then we may also declare that it is valid for x = 0 (for we cannot do 0 / 0). Ruffini's Rule suggests that the user sets up a division, *y* / *x,* but when we look at the proof why it works, [6] then we see addition and multiplication, and thus division (or repeated subtraction) is only an *interpretation*. The method works on the coefficients, and it isn't for nought that the term "synthetic division" is used.

The slope of a curve Δ*f* / Δ*x* contains the notion of division (or ratio). See also the definition of *tangent* in trigonometry for a right-angled triangle. This notion of a slope generates the link between derivative and integral, as the integral uses (Δ*f* / Δ*x*) * Δ*x* to find Δ*f.* The fundamental theorem of calculus is: A function gives the area under its derivative. [7]

A crucial insight:

> When we want to find the root factor x + 1 in (x^2 – 1), then we don't have to assume x ≠ -1, but we can assume the unrelated x ≠ 1, and then isolate the root factor as (x + 1) = (x^2 – 1) / (x – 1).

One might deem this acceptable. It might be a rationale for the theory of "rational function" – group theory version (RF-GT) to define such singularities away. This theory doesn't seem to care that we must also say something about factor x + 1 at x = 1, but it would be straightforward to plug those holes in multiplicative form. The key question then is:

> If we are willing to assume x ≠ 1 and adjust the domain afterwards (in multiplicative form) to again include it, then why would we not do so for x ≠ -1 directly ?

Reasoning like this generates the notion of the dynamic quotient as a useful extension of our vocabulary.

Students must simplify algebraic expression like (x^2 – 1) / (x – 1) anyhow. Since the dynamic quotient allows them to do so consistently with (x^2 – 1) // (x – 1), there is no reason not to allow them to do so for the derivative too.

Eliminating factors is one way of simplification. There might be more ways. Thus the dynamic quotient uses the general notion of "simplification".

[6] https://en.wikipedia.org/wiki/Horner's_method#Description_of_the_algorithm
[7] http://mathworld.wolfram.com/FundamentalTheoremsofCalculus.html

3.4. Perspective on division

The core of the new algebraic approach to the derivative lies in a new look at division. While division is normally defined for numbers, we now use the extension with variables and expressions with variables. Variables have their domains. By default the domain is the real numbers. (There might be symbols with unspecified (only potential) domains though: the "nondetermineds" of RF-GT.) Thus, while Descartes, Fermat, Newton and Leibniz didn't have Cantor's set theory, we now use this to replace a bit of Analysis by Algebra. (Instead as happens in the theory of rational functions, that Cantor's set theory is used to widen the gap.)

Let us distinguish the passive division result (noun) from the active division process (verb). For didactics it is important to write y for the numerator and x for the denominator, and not the other way around. In the active mode of dividing y by x we may first simplify algebraically under the assumption that $x \neq 0$, or that 0 is not in the domain of the denominator. Subsequently the result can also be declared valid for $x = 0$. This means *extending the domain*, i.e. not setting $x = 0$ but merely including that element in the domain.

Active division is not an entirely new concept since we find the main element of simplification well-defined in the function Simplify in *Mathematica,* see Wolfram (1996). For us there is the particular application of Simplify[y / x]. This doesn't claim that this well-definedness satisfies conditions for RM. For empirical research, it removes ambiguity, where students will have various levels of skills on simplification, and we can refer to the computer output as an empirical standard. The active notion of division still requires a separate notation for our purposes. Denote it as y // x or ($y\ x^D$) where the brackets in the latter notation are required to keep y and x together, and where the D stands for *dynamic division*. In the same line of thinking it will be useful to choose static $H = -1$, and have $x\ .\ x^H = 1$ for $x \neq 0$. H gives a half turn as imaginary number i gives a quarter turn.

There is already an active notion (verb) in taking a ratio $y : x$. But a ratio is not defined for $x = 0$. Normally we tend to regard division y / x as already defined for the passive result without simplification – i.e. defined except for $x = 0$. Non-mathematicians will tend to take y / x as an active process already (so they might denote the passive result as y // x instead). For some it might not matter much, since we might continue to write y / x and allow both interpretations depending upon context. This is what Gray & Tall (1994) call the "procept", i.e. the use of both concept and process: "The ambiguity of notation allows the successful thinker the flexibility in thought (...)". In that way the paradoxes of division by zero are actually explained, i.e. by confusion of perspectives. It seems better to distinguish y / x and y // x.

3.5. Already used in mathematics education

Clearly, mathematics education already takes account of these aspects in some fashion. In early excercises pupils are allowed to divide 2 a / a = 2 without always having to specify that a must be nonzero. At a certain stage though the conditions are enforced more strictly. A suggestion that follows from the present discussion is that this process towards more strictness can be smoother by the distinction between / and //.

An expression like $(1 - x^2) / (1 - x)$ is undefined at $x = 1$ but the natural tendency is to simplify to $1 + x$ and not to include a note that there is branching at $x \neq 1$, since there is nothing in the context that suggests that we would need to be so pedantic, see **Table 2**, left column. This natural use is supported by the right column. The current practice in teaching and math exams is to use the division y / x as a hidden code that must be cracked to find where $x = 0$, but it should rather be the reverse, i.e. that such undefined points must be explicitly provided if those values are germane to the discussion. Standard graphical routines also tend to skip the undefined point, requiring us to give the special point if we really want a discontinuity.

Table 2. Simplification and continuity

Traditional definition overload	*With the dynamic quotient*
$f(x) = (1 - x^2) / (1 - x) = 1 + x \quad (x \neq 1)$ $f(1) = 2$	$(1 - x^2) // (1 - x) = 1 + x$

In common life there is no need to be very strict about always writing "//". Once the idea is clear, we might simply keep on writing "/" as a procept indeed. It remains to be tested in education however whether students can grow sensitive to the context or whether it is necessary to *always* impose strictness. For the mathematically inclined pupils or students graduating at highschool one would obviously require that they are aware that y / x is undefined for $x = 0$ and that they can find such points.

3.6. Subtleties

The classic example of the inappropriateness of division by zero is the equation

$$(x - x)\,(x + x) = x^2 - x^2 = (x - x)\,x,$$

where unguarded "division" by $(x - x)$ would cause $x + x = x$ or $2 = 1$.

This is also a good example for the clarification that the rule, that we should never divide by zero, actually means that we must distinguish between:

- *creation* of a quotient by the choice of the *infix* between $(x - x)\,(x + x)$ and $(x - x)$
- *handling* of a quotient such as $(x - x)\,(x + x)$ *infix* $(x - x)$ once it has been created.

The first can be the great sin that creates such nonsense as 2 = 1, the second is only the application of the rules of algebra. In this case, $x - x$ is a constant (0) and not a variable, so that simplification generates a value Indeterminate, for both infices / and //. (One may notice that $x - x = 0$ is the zero polynomial $Q[z] = 0$ in the reference to RF-GT above.)

Also $(a\,(x + x) / a)$ would generate $2x$ for $a \neq 0$ and be undefined for $a = 0$. However, the expression $(a\,(x + x) // a)$ gives $2x$, and this result would also hold for $a = 0$, even while it then is possible to choose $a = x - x = 0$ afterwards: since then it is an instant (and not presented as a variable).

3.7. The derivative

The algebraic definition of the derivative then follows directly:

$$f'[x] = \{\Delta f // \Delta x, \text{ then set } \Delta x = 0\}$$

This means first algebraically simplifying the difference quotient, expanding the domain of Δx with 0, and then setting Δx to zero.

The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and its Cauchy shorthand $\lim(\Delta x \to 0)\ \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken. Instead, using $\Delta f // \Delta x$ on the formula and then extending the domain with $\Delta x = 0$, and subsequently setting $\Delta x = 0$ is not paradoxical at all. Students only need an explanation why one would take those steps.

Much of calculus might well do without the limit idea and it could be advantageous to see calculus as part of algebra rather than a separate subject. This is not just a didactic observation but an essential refoundation of calculus. E.g. the derivative of $|x|$ traditionally is undefined at $x = 0$ but would algebraically be sign$[x]$, see Colignatus (2011b). The derivative gives the change in the area under the curve, and this might not be the same as the slope of the incline (tangent line).

3.8. Differentials

There is the following progress from 2011 to 2016:

- COTP (2011ab) uses "***d**f* / ***d**x*" as a icon only, or "***d*** / ***d**x*" as an operator, to link up with history only, so that everyone who still uses this notation for the derivative can see that this has the same outcome,
- Colignatus (2016d) proposes to use dx and dy as variables, and to define $dy = f'[x]\ dx$ so that $dy // dx = f'[x]\ dx // dx = f'[x]$. This is actually the situation with the ray that this section started out with. Thus the derivative $f'[x]$ is found by other means, and then is used to set up the ray with dx and dy. The dynamic quotient $dy // dx$ should not be confused with finding the derivative (since dy is defined by using the derivative).

For users new to the notions of the dynamic quotient and the algebraic approach to the derivative, the relation $dy // dx = f'[x]$ might be confusing since they might think that the dynamic quotient suffices to find the derivative, without the need to set $\Delta x = 0$. (An answer to this is: There are various roads to Rome but only few ways to build it. Check again what the proper definition of the derivative is.)

3.9. Derivative at a point $x = a$

In the standard notation for the derivative, x is fixed and the new variable is Δx.

There is also a notation when x is retained as a variable, and the fixed value is $x = a$. If we want to find the derivative at a point $x = a$ then we would use above method to find $f'[x]$ and then substitute the value to find $f'[a]$. This suffices.

If one wishes to specify a in the deduction, then use:

$$\{\ (f[x] - f[a]) // (x - a), \text{ then set } x - a = 0\ \} = f'[a]$$

The following notation would be advised against, since it mixes changes of perspectives:

$$\{\Delta f // \Delta x, \text{ then set } \Delta x = x - a = 0\} = f'[a]$$

NB. An angle on didactics:

The form $(f[x] - f[a]) // (x - a)$ may be more agreeable to students than $(f[x + \Delta x] - f[x]) // \Delta x$. An intermediate solution is already to use $(f[x + h] - f[x]) // h$. There is $(f[b] - f[x]) // (b - x)$, that after resolution still can be evaluated at $x = b$ to find the derivative $f'[x]$ and then be evaluated at $x = a$. However, the use of Δx relates to "difference" and "***d**f* / ***d**x*" and has the advantage that there is no subtraction $b - x$. Potentially, it might be better didactics to present the notations alongside to each other, which the invitation to students to use the notation that they like, so that they are encouraged to look deeper into this.

4. Definition of the incline

I will use the word "incline" instead of "tangent (line)" since the incline may also cut the function, see Colignatus (2016e). Let us use "tangent" in trigonometry only.

The core notion is that the slope s must be taken as the *slope of the curve* at the point of consideration. We don't have just the line. First we determine the slope of the curve, and then create the incline with it.

The point-slope form with Δx is: $y - f[x] = s\ \Delta x$ at the point of inclination $\{x, f[x]\}$.

The point-slope form with a is: $y = \text{incline}[x] = s\ (x - a) + f[a]$ at the point of inclination $\{a, f[a]\}$.

The standard form is $y = c + s\ x$, with slope is s and constant $c = f[a] - s\ a$.

5. Comments w.r.t. Shen & Lin (2014)

5.1. Need to get rid of limits

I tend to agree with their following statement (p2) why we should be looking for better ways to teach calculus:

Calculus is one of the most important tools in a knowledge based society. Millions of people around the world learn calculus everyday. All engineering, science, and business major undergraduate students must take calculus. Many high schools offer calculus courses. The usefulness and power of calculus have been well recognized. Nonetheless, calculus is a mysterious subject to many people and is regarded by the general public as accessible only to a few privileged people with special talents. Tight schedules and high fail rates for the first semester calculus have given the course a reputation as a monster, a nightmare, or a phycological barrier for many students, some of whom are even STEM (science, technology, engineering and mathematics) majors. Calculus can be a topic that causes people at a social gathering to shake their heads in incomprehension, shy away from the daunting challenge of understanding it, or express effusive exclamation of awe and admiration. It is also sometimes associated with conspicuous nerdiness. In classrooms, the student-instructor relationship can be tense. Some students regard calculus instructors as inhuman and ruthless aliens, while instructors frequently joke about students' stupidity, clumsiness, or silly errors. Tedious and peculiar notations coupled with fiendish and complex approaches to calculus teaching and learning may have contributed to the above unfortunate situation.

A major cause of this mystery and scare of calculus is unnecessarily complex terminologies, including definite and indefinite integrals, derivatives defined as limits, definite integrals defined as limits, the difference between dx and Δx, and using ever-finer divisions of an area under a curve to approximate a definite integral (i.e., introducing the definite integral by area and limit), to name but a few. Additionally, the Riemann sum with its arbitrary point x_i^* in the interval $[x_i, x_{i+1}]$ complicates calculation procedures and adds more confusion. These conventional concepts and notations are essential for professional mathematicians who research mathematical analysis, but are absolutely unnecessary to the majority of calculus learners, who are majoring in engineering, science, business, or other non-mathematical or non-statistical fields.

5.2. Simultaneous introduction of derivative and integral

COTP (2011) also adopts the simultaneous introduction of derivative and integral. It is interesting so see that professor Lin already does so for over two decades (p3).

2011), rather than indirectly uses the secant line method with a limit. We introduce derivatives and antiderivatives simultaneously, using derivative-antiderivative (DA) pairs. Our introduction to integrals is directly from the DA pairs and the height increment of an antiderivative curve. The height increment approach has been advocated by Q. Lin in China for over two decades (see Lin (2010) and Lin (2009) for two recent examples). We define the area under the curve of an

Some comments are:

(a) One has to start somewhere. COTP starts with the area (multiplication) but immediately refers to the counterpart that a change in area provides a slope (division), such that it is clear that these concepts are directly related.
(b) These notions are relative indeed: a relation between two functions.
(c) Indeed, an area starts somewhere and ends somewhere: the definite integral.
(d) But the function itself has no such boundaries: the primitive or indefinite integral.
(e) Thus there is the notion that a function gives the area under its derivative.
(f) COTP (2011) uses the term "surface" to link up with the S of the integral sign (Sum). There might be a tension from the distinction between "surface" (notion) and "area" (measurement), but there is an advantage of the use of S.

(g) An advantage of the term "integral" rather than the term "surface" or "area" is that it abstracts from the particular notion of geometric area. In some realms the notion of a particular integral might be more complicated than the mere geometric interpretation that it represents the area under its derivative.
(h) I prefer the term "primitive" and am not in favour of the term "antiderivative" for the indefinite integral, and restricting the term "integral" to the definite integral. It should suffice to say that integral and derivative are two faces of the same coin. Some authors might perhaps distinguish between "antiderivative" and "integral" by means of the constant of integration, but my preference is to follow Eric Weisstein of MathWorld, except that I would prefer that he replaces "antiderivative" with "primitive (antiderivative)". [8]

5.3. The definition of area has already been given

COTP (2011) calculates the area under the curve by conventional means, and then introduces the name "integral" for the function that gives the area.

I don't think that one can do the reverse, as Shen & Lin (2014) are suggesting. The notion of area is already defined, notably be the (unit) square. Thus it is better to show that "integral" is only another name for a proper calculation.

> Lin (2010) and Lin (2009) for two recent examples). We define the area under the curve of an integrand by the integral, and then explain why the definition is reasonable. This is a reversal of the traditional definition, which defines an integral by calculating the area underneath the curve of an integrand.

However, COTP (2011) only shows a calculation by conventional means for constant and line. It doesn't do so for higher polynomials or exponential number or trigonometry. The approach is: Assume that *f* gives a surface under some unknown *g,* and then find *g.*

In that manner there is no need to create a Riemann sum to show that the surface under some given *g* generates a function *f.*

Thus, don't define the integral as giving the area, but start with a function that gives an area.

(Professor Lin with his reference to height increment *f*[*x*] – *f*[*a*] is also close to this idea.)

5.4. Notation

Notations like *f* '[*x*] and Sur[*f*[*x*], *a, b*] are indeed sufficient, certainly for a two hour introductory course.

However, for a longer course, students must be told about historical notations that are still conventional. There is no need to use historical notations in sums, but students must be informed.

I would prefer Sur[*f*[*x*], *a, b*] above *I*[*f*[*x*], *a, b*] for this introductory phase, but agree that *Mathematica's* Integrate[*f*[*x*], {*x,* a*, b*}] is fine for general application.

> We also demonstrate that an introduction to the basic ideas of calculus does not need to use many complicated notations. Thus, the notations of derivatives $f'(x)$ and integrals $I[f(x), a, b]$ in this paper are parsimonious, simple, computer friendly and non-traditional.

5.5. Definitions of tangent, incline and double root line

Remarkably, Shen & Lin (2014) start with the slope (p3) and then show the double root interpretation (for a parabola), and they refer to various authors including Range (2011).

[8] http://mathworld.wolfram.com/ConstantofIntegration.html

Problematic is this statement:

The slope of a curve at a given point is defined as the slope of a tangent line at this point. Figure 2 shows three points: P, A and B. T_P represents the tangent line at P whose slope is defined as the slope of the curve at P. The tangent line's slope is used to measure the curve's steepness. Calculus studies (i) the slope $\tan\theta$ of a curve at various points, and (ii) the height increment H from one point to another, say, A to B, as shown in Figure 2. Our geometric intuition indicates that height H and slope $\tan\theta$ are related because H increases rapidly if the slope is large for an upward trend. A core formula of calculus is to describe the relationship between these two quantities.

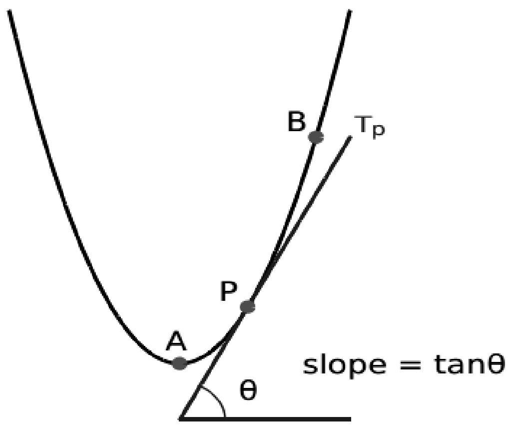


Figure 2: A curve, three points and a tangent line without coordinates.

(1) Conventionally, the tangent is defined by the slope of the curve.
(2) Shen & Lin (2014) are inconsistent in their their statements.
    (a) In the first sentence: Shen & Lin (2014) reverse "causality" by defining the slope of the curve by "the slope of the tangent".
    (b) In the third sentence they return to the conventional approach: "*Tp* represents the tangent line at *P* whose slope is defined as the slope of the curve at *P*."
    (c) In the fourth sentence they return to the first sentence (c): "The tangent's line's slope is used to measure the curve's steepness."
(3) I was struck by the two last sentences.
    (a) I have used the graph, though with more detail, as Murray Spiegel's graph [9] reproduced in Colignatus (2016d).
    (b) I have used Δf and Δx to point to the connection to the slope.
    (c) I have used Δf and Δx to point to the connection between integral and derivative, namely as $\Delta f \approx f\,'\,[x]\,\Delta x$.
    (d) But I haven't used this geometric image yet to point explicitly to the connection between derivative and integral.
    (e) Thus it is an important step (set by Shen & Lin here) to call attention to the issue that this graph also shows the close connection between derivative (slope) and integral (height increment).
    (f) NB. As explained in the overview Table 1, COTP uses the trick that the primitive F already gives a surface (area) under some function f that however needs to be determined. The term "height increment" for ΔF thus translates directly as "area increment". See COTP:154 for the graphs for $\{x, F\}$ and $\{x, f\}$.
    (g) Thus a suggestion for didactics is to use this type of graph in this manner too. Or introduce the students to the notions, then develop the formulas, and then to return

[9] https://boycottholland.files.wordpress.com/2016/12/1962-murrayspiegel-p58-fig4-1.jpg?w=545&h=554

to the graph and discuss: "What we see here, is also the close connection between derivative and integral." And this is precisely what Shen & Lin (2014:17) do (their figure 11).

(4) Subsequently, on this page, Shen and Lin (2014) explain that "*m* is the slope to be determined by the condition that that the tangent line touches the curve at one point. In fact, "tangent" is a word derived from Latin and means "touch"."
   (a) Here *m* is the slope of the line, thus the double root line, and this doesn't give the tangent in conventional definition that uses the slope of the curve.
   (b) The tangent can also cut and not just "touch" a curve.

5.6. Proof status

Shen & Lin (2014) use the phrase "We claim that the slope of the curve ...". Thus they are well aware that they only claim something here. There is no proof that the double root line also gives the slope of the curve. There is a section on rigour below.

PM. There is no link to the earlier formula for the tangent in trigonometry, other than the graph. The graph is intuition only. If various student were given the same curve and point, many would draw (slightly) different slopes.

5.7. The term "antiderivative" for indefinite integral and using "integral" only for the definite integral

Europe uses "primitive" and in the USA apparently the term "antiderivative" has come up. [10] Shen & Lin (2014) follow the USA. I find the terms "primitive" and "derivative" more natural. It fits COTP's start with surface, though this might be a coincidence.

Definition 1. (Definitions of derivative as slope and DA pair). *The slope $2x$ is called the derivative of x^2. Further, x^2 is called an antiderivative of $2x$. And $(2x,\ x^2)$ is called a derivative-antiderivative (DA) pair.*

For a general function $y = f(x)$, the slope of the curve $y = f(x)$ is called *derivative* and is denoted by $f'(x)$ or y', and $f(x)$ is called antiderivative of $f'(x)$. Thus, $(f'(x), f(x))$ is a DA pair.

Shen & Lin (2014) later reserve the word "integral" for the "definite integral" but this causes problems for student when they link up with the literature.

PM. However, there appears to be a real problem, as the authors state on p2:

In the first semester calculus, instructors repeatedly emphasize that an indefinite integral yields a function while a definite integral yields a real value. When an instructor discovers that some students still cannot tell the difference between a definite integral and an indefinite integral in the final exam, s/he becomes disappointed and complains that these terrible students did not pay attention to her/his repeated emphasis on the difference, but s/he rarely questions the necessity of introducing the concept of indefinite integrals and their notations.

Perhaps this might be solved by always writing "definite integral (area)" and "primitive or indefinite integral (function)" in these introductory texts.

5.8. Interpretation as a slope

Shen & Lin (2014:5) provide this argument to interprete the derivative as a the slope of the curve.

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[10] http://math.stackexchange.com/questions/47703/how-common-is-the-use-of-the-term-primitive-to-mean-antiderivative

If $f(x) = C$ is a constant, then $y = C$ represents a horizontal line whose slope is 0 for any x. Hence $(C)' = 0$, and $(0, C)$ is a DA pair.

If $f(x)$ is a linear function, then $y = \alpha + \beta x$ represents a straight line whose slope is β for any x, hence $(\alpha + \beta x)' = \beta$, i.e., $(\beta, \alpha + \beta x)$ is a DA pair.

Thus, the derivative's geometric meaning is the slope of the curve $y = f(x)$: $f'(x)$ is large at places where the curve $y = f(x)$ is steep. At a flat point, such as the maximum or minimum point of $f(x)$, the slope is zero since the tangent lines at these points are horizontal.

This direct interpretation only works for constant and line. I don't think that Shen & Lin intend to generalise this in such manner, but a student might think that this could be done in this manner, and it is better to avoid this. The interpretation for constant and line are useful however for inspiration to find a general statement (as COTP does).

5.9. Height increment and integral

Shen & Lin (2014:7-8) link the height increment to the definite integral, i.e. the area over an interval.

This is in fact how COTP starts with its exposition.

But observe the consequences in didactics:

- My impression is that the Shen & Lin order of first derivative and then integral introduces a confusion between the vertical increment, that is relevant here, and the horizontal increment, i.e. the interval [*a, b*], that is not relevant here. Why would the student be interested in this particular interval at this moment in the lesson ?

- In the order taken in COTP there is a natural interest in the interval [*a, b*] at the start of the discussion. The term "integral" derives from the "integration over an interval" and thus the multiplication with the horizontal *dx* is important. Yet for the integral the vertical *dy* is more important, as the discussion continues with that.

3 Height increment and integrals

When we trace a curve, we care about not only the slope, but also the ups and downs of the curve, i.e., the increment or decrement of the curve from one point to another. When we drive over a mountain road, we also care about both steepness (i.e., slope) and elevation. Apparently, the slope and height increment are related. The slope has already been defined as derivative in the above section. In this section, the height increment is defined as *integral*, since the height increment or elevation increment is an integration process, or an accumulation process of point motion, measured by both speed and time.

7

For a function $y = f(x)$, its increment from $A(a, f(a))$ to $B(b, f(b))$ is $f(b) - f(a)$ as shown in Figure 3. Another notation for the increment is $f(b) - f(a) = f(x)|_a^b$. This height increment is used to describe the integral definition below.

Definition 2. (Definition of integral as height increment of a curve). *The function's increment $f(b) - f(a)$ from $A(a, f(a))$ to $B(b, f(b))$ is defined as the integral of the derivative function $f'(x)$ in the interval $[a, b]$ and is denoted by $I[f'(x), a, b] = f(b) - f(a)$. Here, $f'(x)$ is called the integrand, and $[a, b]$ is called the integration interval.*

Example 1. Given $f(x) = x$, $f'(x) = 1$, and $[a, b] = [0, 2]$, we have

$$I[f'(x), a, b] = I[1, 0, 2] = x|_0^2 = 2 - 0 = 2. \tag{17}$$

The above still is fairly complex (even without limits). We have *f, f ', a, b, two points A, B, and a new notation I*[....].

In COTP the problem doesn't arise. The discussion starts with surface or area, whence the multiplication of horizontal and vertical intervals is available from the start in a natural fashion. The subsequent steps are to introduce the primitive-derivative pair, and then return to the original issue, and reformulate the surface or area in terms of these new notions as a definite integral.

5.10. Mean Value Theorem

Shen & Lin (2014:12) clarify that their paper is only a didactic exposition, of how calculus can be presented to students without the use of limits, but, that proper theory may still require limits. They don't intend to present an essential redesign of calculus.

Rigorous mathematics for MVT would require one to prove the above statement "$(f(b) - f(a))/(b - a)$ must meet the mid-ground slope $f'(c)$ at one point c in $[a, b]$," namely, it proves the existence of the point c. This is to prove the intermediate value theorem and is beyond the scope of this introductory lecture.

5.11. Definite or indefinite

Shen & Lin (2014:12):

> Therefore, the integral $I[f'(t), a, x] = f(x) - f(a)$ is the increment of the antiderivative from a to x, and is also the area for the region between the integrand derivative function and $y = 0$ in the interval $[a, x]$, i.e., the region bounded by $y = f'(t), y = 0, t = a$ and $t = x$. The traditional definition of an integral is from the aspect of an area that is defined as a sum of many rectangles of increasing narrow widths, under the condition of each width approaching zero. For non-mathematics majors and general public, the condition of each width approaching zero, which is a concept of limit, adds complexity and confusion to the traditional definition of integral. In contrast, the geometric meaning of our direct integral is the height increment of the antiderivative, not the area underneath of the derivative. The area is only regarded as an additional geometric interpretation according to the intermediate value theorem. Under this interpretation of area, we have the following example.

(a) I agree that the sum of rectangles can be avoided, and that there is an approach that is equally exact.
(b) The term "direct integral" is in use for other applications. [11] It suffices to refer to the distinction between definite (area) and indefinite (function) integral.
(c) The distinction between height increment Δf and area under the derivative is false, since f gives the area under its derivative, and the height increment is an area increment.
(d) The authors mean to say that they want to put didactic emphasis on Δf rather than on the calculation of the area (via Riemann sums) (as so often happens in calculus).

5.12. Mean Value Theorem for definition of the derivative

Shen & Lin (2014:13):

> Calculating the slope using the factorization method works for polynomial functions, but the procedure is tedious. The procedure may not even work for transcendental functions like $y = \sin x$. The MVT provides another way to calculate the slope by using the slope of a secant line. In the above MVT, if B moves very close to A, then the mean value $(f(b) - f(a))/(b - a) = f'(c)$ in Theorem 1 is approaching the slope at A, since c is approaching a, forced by b approaching a. The formal writing is
>
> $$\lim_{b \to a} \frac{f(b) - f(a)}{b - a} = f'(a). \tag{27}$$
>
> This can also be considered a definition of derivative and is called *defining a derivative by a limit*. This procedure is efficient to calculate the derivative by hand and to derive many traditional derivative formulas in the earlier years of calculus.

(a) I agree that polynomial factorisation quickly becomes tedious.
(b) I can understand that there are some attractive features in developing the intuition of the double root line, yet, this causes such factorisation, and has the "excess burden" of the polynomial remainder theorem and so on, while eventually one still has to return to limits to find the slope of the curve. (Leaving aside what COTP also finds on exponential function and trigonometry.) Thus in sum I don't quite understand why one would adopt the didactics with the double root line.
(c) Who wishes to emphasize the simultaneous relevance of derivative and integral (Shen & Lin (2014) and COTP (2011)), would focus attention on the slope of the curve, given by the tangent in trigonometry, and not on the double root solution.
(d) If Shen & Lin (2014) are willing to invoke the Mean Value Theorem anyhow, then why not do so to start with ? By this is deduction and rationality, and not student psychology.

[11] https://en.wikipedia.org/wiki/Direct_integral

(e) The claim that limits are "efficient" to derive results by hand is dubious. It took brilliant mathematicians some effort e.g. to work out the outcomes for the exponential function and trigonometry. The steps can now be explained and math-savvy students would be able to follow these steps, yet, for common practice we would want them to work efficiently by applying the general rules, like (*e*^*x*)' = *e*^*x* and so on.

(f) A key element in the derivation by limits remains the underlying algebra. Traditionally, one first determines an algebraic solution, and then puts a limit around this to invoke a solution for the singularity. The working horse is the algebra and not the limit.

(g) We aren't only looking at efficiency. We are also looking at good didactics, so that students know what these formulas and graphs stand for. There are some suggestions in COTP (2011) that Shen & Lin (2014) haven't looked at yet.

Appendix A. From the comparison of WIC Prelude and COTP

The following is from the "Introduction" in Colignatus (2017e), the comparison of WIC Prelude and COTP.

A disclaimer is that I only have read Range (2011) (2014) and (2016bc), namely targetted at the comparison on the algebraic approach.

The approaches by Range (2011) (2014) (2016bc, Prelude) and Colignatus (2007, ALOE) (2011, COTP) have in common that they avoid limits and rely on algebra, both with the claim that they do not sacrifice rigour and understanding. There is a key difference though:

- The set-theoretic approach to functions (Bourbaki) takes $f = \{\{x,y\} \mid x$ and y in their sets$\}$. The x and $y = f[x]$ are *elements* and not quite *variables.* For *analysis*, the sets are (real) numbers. This necessitates notions of *numerical* continuity and limits. Range adopts this setting for the full book (2016a), already with the exponential number. In the "Prelude" (2016c), the reference to algebra and polynomials is ("only") a didactic tool for lowering the (conceptual) barrier for students, and to link up to the history on notions of tangency. Thus Range doesn't claim a fundamental redesign of calculus. His claim on the avoidance of limits only concerns the derivation of the double root line. Limits are still required to show that this also generates the slope of the curve.
- Historically, a function was a *proscription* of how to turn an input into an output. This generated some study of notations and algorithms, see also Cha (1999), as we see nowaday in *computer algebra,* with the algebra of *variables* and *expressions.* Colignatus (2007) (2011ab) rekindles this approach. Information about the function is contained in its expression. There is a notion of "continuity in form". This information can be used when particular methods of arithmetic generate problems, notably with arithmetic division at zero. We can define a notion of "dynamic quotient" that manipulates the domain. This dynamic quotient allows an algebraic definition of the derivative. An algebraic approach is also possible for the exponential number and trigonometry. Looking at calculus in algebraic manner again would be a fundamental redesign, after the Cauchy and Weierstrasz turn to numbers. The approach originated from didactics and it would be up to mathematicians to see how far the redesign can be developed further. Obviously, students majoring in mathematics would have to know both methods.

Table 3 gives the overview of the difference in approach.

Table 3. Overview of the difference

	Limits	*Algebra*	
		"Algebraic functions" [12]	*Expressions*
Double root		Range (2016c:32) proof but no relation to slope	Sometimes handy [13]
Slope	Standard Range: non-algebraic functions Colignatus: majors in mathematics only	Range (2016c:5): The slope of the curve isn't in the definition of "tangent"	Colignatus (2011:154 & 224): "dynamic quotient" allows to set $\Delta x = 0$ for the derivative (i.e. setting and not just a limit)
Tangent	Definition by trigonometry	Definition by double root	Definition by trigonometry

On content, I agree with the actual deductions and many observations by Range. A main comment is that I would add that the dynamic quotient allows a fundamental redesign of calculus. Range's contribution lies in didactics, yet his approach in didactics does not convince me. Obviously, it are the students who must tell what didactics works for them. Thus I am looking forward to classroom experiments, i.e. not just usage but randomised controlled trials. The following comments should be helpful to determine what to check.

The major problem in didactics is that Range *redefines* the *tangent* as the "double root line", i.e. the line that "overlaps with the curve and has a double root" (summary, no quote).

- In the common vocabulary, the tangent by definition takes the slope of the curve. This slope is found by means of other criteria.
- In this redefined "tangent" there is no reference to the slope of the curve, whence the notion of derivative in relation to the integral of the curve disappears. [14]
- This better be called the "double root line" (my phrase, not in Range's vocabulary).
- The use of the common phrase "find the tangent" asks for the slope of the curve, and doesn't ask for the "double root line". Thus the redefinition of "tangent" create confusion w.r.t. to the accepted vocabulary. A teacher used to the common vocabulary might think that one asks for the slope of the curve but the student of Range (2016c) will only generate the line with the double root.
- We need an additional theorem that the double root line also generates the slope of the curve (...). This also holds when one redefines "tangent" to become the double root line (since there is nothing in the notion of a double root line that directly links to the slope of the curve).

In supplement to this copy from the "Introduction", the following can be mentioned.

Range (2016c:32) contains a (crucial) proof that in case of *factorisation* with some $q[x]$: there is a double root iff $m = q[a]$. Then the continues:

[12] Range (2016c:34) proves that all algebraic functions are continuous, which proof uses a limit for the notion of continuity. This continuity is essential for Range's results. Thus the claim that his method doesn't use limits is dubious. The double root line can be found without limits but the trigonometric tangent required for the slope of the function still relies on it.
[13] Since the error function $\varepsilon[x, \Delta x]$ contains unknown slope *s,* algebraic methods like comparing coefficients may generate solutions for *s* faster than standard deductions, but one still would need a link to the slope of the curve. Thus, a method of calculation should not be confused with a fundamental relationship.
[14] In addition, there is only the avoidance of limits if one accepts algebraic methods to isolate factors of polynomials (which methods don't use limits).

Geometrically, this means that the line described by the linear function $y = f(a) + m(x - a)$ intersects the graph of $y = f(x)$ at $(a, f(a))$ with multiplicity at least two if and only if $m = q(a)$. Consequently, the line given by $y = f(a)+q(a)(x-a)$ is *the tangent* to the graph of f at $(a, f(a))$.

The reader who is used to the conventional notion that tangency gives the slope of the curve, finds this miraculous, since how does this "iff" statement translate in the "consequence" that *q*[*a*] is the slope of the curve ? This reader then must backtrace the argument till the definition of tangency, where it appears that Range has changed this from the conventional notion into the property that this line overlaps with the curve with double root. Thus Range's "Consequently" is correct in his vocabulary, but the traditionally minded reader still requires a proof that links the double root line to the slope of the curve (the tangent in trigonometry).

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