TRIG RERIGGED 2.0

Trig Rerigged 2

Trigonometry in primary and secondary education

Thomas Colignatus

Samuel van Houten Genootschap

Colignatus is the preferred name of Thomas Cool in science. He is an econometrician (Groningen 1982) and teacher of mathematics (Leiden 2008). http://thomascool.eu, cool at dataweb.nl

September 4 & 7 2016, draft

Supported by

Samuel van Houten Genootschap
Wetenschappelijk bureau van het Sociaal Liberaal Forum
Scientific bureau of the Social Liberal Forum
The Hague

ISBN ...
Published by ...

American Mathematical Society: MSC2010
00A35 Methodology and didactics
(97-XX Mathematics education)
97F02 Arithmetic, number theory ; Research exposition
97G70 Analytic geometry. Vector algebra
97H20 Elementary algebra

Journal of Economic Literature: JEL
P16 Political economy
I20 General education
A20 Economics education

Nederlandstalige Uniforme Rubrieksindeling: NUR
841 Onderwijskunde
918 Wiskunde algemeen
780 Economie en bedrijf algemeen (Onderwijs van wiskunde en beheer ervan, Political Economy)

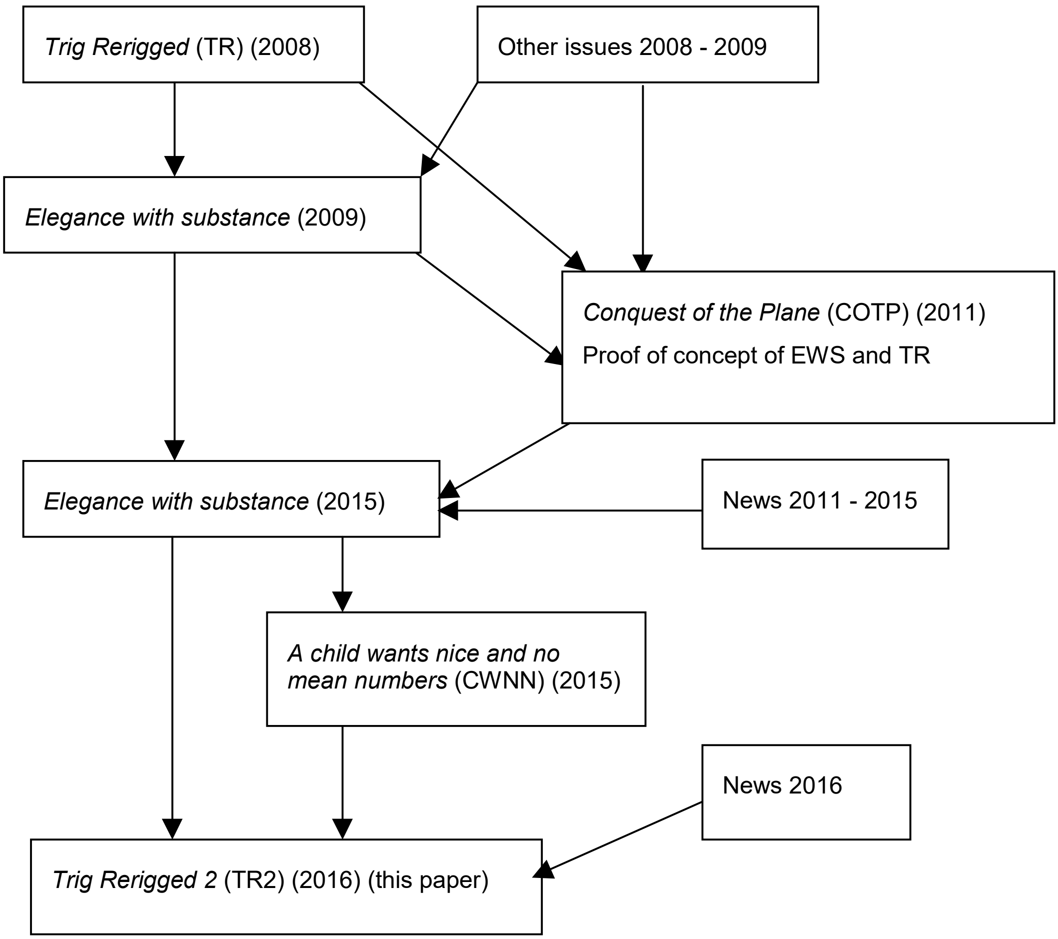
Prologue

This text is very preliminary. On August 30 I had the insight on page 15 with graphs on page 16 below, and it is now only September 4. Perhaps I am still trying to discover what the insight actually was. September 7: A summary is in Colignatus (2016c).

The text might never become a booklet but only adopts the form of a booklet because this fits with the books (with pdfs onlline):

- *Elegance with Substance* (EWS) (2009, 2015)
- *A child wants nice and no mean numbers* (CWNN) (2015)

The following shows how the different books and papers relate to each other. *Trig Rerigged* (2008) and other issues caused EWS (2009). *Conquest of the Plane* (COTP) (2011) was a proof of concept. The 2nd edition of EWS (2015) came with a companion for elementary school CWNN. The news in 2016 causes TR2.



Contents in brief

Introduction **9**

Archi or Pi (EWS:27) **10**

Trig rerigged (in COTP) (EWS:64-65) **11**

Circles and measurement of angles (CWNN:99-100) **13**

Insight and design **15**

Introduction of angles in primary education **16**

Towards the end of primary education **18**

Xur and Yur in primary education ? **19**

Highschool **22**

Acknowledgement and protest **23**

Conclusions **25**

Literature **26**

Index **27**

Contents

Introduction **9**
Notation of fractions 9
Two approaches and a new synthesis on trigonometry 9
Circumference and surface 9

Archi or Pi (EWS:27) **10**

Trig rerigged (in COTP) (EWS:64-65) **11**

Circles and measurement of angles (CWNN:99-100) **13**

Insight and design **15**

Introduction of angles in primary education **16**
Hook disk 17
An issue of design 17

Towards the end of primary education **18**

Xur and Yur in primary education ? **19**
Hook function 19
Domain and range 20
Graphs and dynamic displays 20
Calculations 20

Highschool **22**

Acknowledgement and protest **23**

Conclusions **25**

Literature **26**

Index **27**

Introduction

This discussion is top-down, and targeted on teachers.

When teachers agree on the approach then it will be useful to develop the bottom-up approach for students, starting in primary school and continuing with secondary school.

For the top-down discussion it is still relevant to give an outline of how the bottom-up approach would look like.

Notation of fractions

Let us take i = quarter turn, H = half turn = i^2 = -1, then $i^3 = H\, i = -i$ = 3 / 4 turn = three per four turns, and $H\, H$ = full turn = 1.

H is an algebraic symbol, pronounced as "eta", and might be substituted with -1 but often is left as it is.

Pronounce 5^*H* as "(one) per 5".

The key property is that $x \,.\, x^H = 1$ for $x \neq 0$. Perhaps CWNN has the nicest introduction to this.

The suggestion is that *H* is used to replace fractions. Thus the deduction $x^H = 1 / x$ is not only superfluous but also counterproductive, as one better learns to work with *H.*

Two approaches and a new synthesis on trigonometry

Trig Rerigged provided an alternative to traditional trigonometry. The news in 2016 is a third approach that gives a synthesis.

The first two approaches are:

(1) Traditional. Use 360 degrees or 2 π radians, and functions sine, cosine and tangent.

(2) Trig Rerigged. Also use the plane itself as the unit of measurement. The unit becomes 1 *unit measure around* (UMA). Also use $\Theta = 2\,\pi$. Thus gives functions *Xur, Yur* and *Tur.* Note that EWS does not propose to eliminate π but only proposes the inclusion of Θ as well. Trig Rerigged looks at trigonometry and then Θ seems more natural.

Before starting with the synthesis, it is better to repeat the relevant sections from EWS and CWNN, that have been reproduced below.

Circumference and surface

This document might be read by students who are not familiar with the formulas for circumference and surface. For the unit circle r = 1, and this gives natural expressions for these parameters: Θ is the circumference and π is the surface.

	Using π	Using Θ
Circumference	$2\,\pi\, r$	$r\,\Theta$
Surface	$\pi\, r^2$	$2^H\, r^2\, \Theta$

Archi or Pi (EWS:27)

The mathematical symbol π (Greek "pi") is defined on a circle as the ratio of the circumference to the diameter. This derives from ancient experience that the measurement of the diameter is very practical. In science and education the diameter has lost its relevance however, and it is the radius of the circle that matters.

Angles are commonly measured in 360 degrees or 2 π radians. Because of the distinction between diameter and radius, education suffers the perpetual factor 2.

It is useful to define Θ (Greek capital theta, pronounced as *Archi* from Archimede) as the ratio of the circumference to the radius. Thus $\Theta = 2\ \pi$.

For angles, we can take the plane itself as the unit of account. One *turn* around the circle is a better measure than 360 degrees.

The advantage of using Θ is twofold:

(1) It is easier to think in terms of whole circles and turns than half circles. As π radians are an angle of 180 degrees, or a half plane, it carries with it a notion of non-completeness. Using Θ / 2 or $2^H\ \Theta$ carries the notion of only a half turn. Indeed, the symbol π is taken from "perimeter" and it has succeeded only half.

(2) There is more outward clarity on the linkage with calculus.
The integral of *x* is ½ x^2 or $2^H\ x^2$.
Thus with radius *r* the circle circumference is $r\ \Theta$ and its surface is ½ $r^2\ \Theta$ or $2^H\ r^2\ \Theta$.
Admittedly, when you look for it then the relationship from calculus can also be seen when using π but the advantage of Θ is that you don't really have to look for it since it tends to stand out more by itself.

Independently, Palais (2001ab) came to the same view (see also his animated website [1]). Palais introduced the three-legged $\pi\!\!\pi$ but this is bound to cause writing and reading errors, let alone confusion, and I remain with Θ.

Here it suffices to point out the mere benefits of using Θ. We will return to trigonometry later on when discussing the measurement of angles, see page 11.

PM 1. Some students confronted with 2 π have the tendency to complete this by applying the calculator and returning 2 * 3.14… = 6.28… With Θ it would be easier for them to stop, and wonder whether the exact Θ is required or the numerical approximation. However, they will meet much of the same problem when they are confronted with Θ / 2 or $2^H\ \Theta$. Hence this issue must be dealt with separately.

PM 2. Rather write $x\ \Theta$ instead of $\Theta\ x$. Current convention is to write $2\ \pi\ r$ but there is advantage in recognizing Θ as an indication of the full turn as a unit of measurement.

2015:

PM 3. There is the tau-community who propose to use tau or $\tau = \Theta = 2\ \pi$. Earlier I considered this option too but rejected it because of the similarity with *r,* the symbol for the radius. This similarity will cause much confusion, especially in handwriting in exams.

PM 4. There is an animation. [2]

---

[1] http://www.math.utah.edu/~palais/cossin.html
[2] https://boycottholland.wordpress.com/2014/07/14/an-archi-gif-compliments-to-lucas-v-barbosa/

Trig rerigged (in COTP) (EWS:64-65)

I am not much of a fan of trigonometry. Apparently I am neither too rational, for the smart way would be to neglect it and proceed with the fun stuff. On the other hand, it was a bad itch that felt like scratching. We already discussed the choice of $\Theta = 2\ \pi$, see page 10. But we can do more.

For students it is a bit confusing that angles are measured counterclockwise. It would be too complex to change this, e.g. also with derivatives. Perhaps there is a moment later on to try it but now we let this rest.

In thinking about angles, people naturally think in turns, half turns, quarter turns. Mathematicians have considered the case, and don't listen. An angle is *defined* as a plane section between two intersecting lines. But it is *measured* (in a dubious distinction with *definition*) with either (a) sine, cosine and tangent, or (b) the arc of the unit circle. A unit circle has radius 1. The circumference can be subdivided in 360 degrees, deriving from the Sumerian measurement of the year and maintained over the ages since 360 is easy to calculate with. Subsequently, it is seen as an "innovation" – the advancement of grade 11 over grade 10 – that said perimeter can also be subdivided in Θ radians.

Most mathematicians would hold that radians and π are dimensionless numbers. For example π would be defined as the ratio of a circumference $2\ \pi\ r$ to the diameter $2\ r$ of any circle. Since numerator and denominator are measured in the same unit of measurement, say the meter, it drops out. I would oppose this, since a 'meter around' is something else that a 'meter in one direction'. Here we have our turns, half turns, quarter turns. The *turn* is its own dimension. But does the meter really disappear ? When we consider a unit circle, then that unit has to be something. Everyone can imagine a circle and also imagine a measuring rod, and each image will be quite arbitrary. But it is curious to argue that this would be without a unit of measurement – precisely since such a measuring rod is imagined too. Thus the unit would be *"unit measure around"* (UMA) and not degrees or radians. For communication it helps to use the already existing unit of measurement, the meter. It is still a choice – and I am inclined to prefer it – to let the UMA be the *"unit meter around"* dimension. When drawing a sine function the student can plot out one meter instead of measuring out $\Theta = 6.28...$ meters. We can also use a circle with a circumference of 1 meter and a radius $r = \Theta^H \approx 16.16$ cm.

We cannot wholly eliminate the unit circle because of sine and cosine and their neat derivatives. Sine and cosine are OK for triangles in arbitrary orientiation too. With co-ordinates, they indicate y and x on the unit circle. Thus let us *call them so* too.

Figure 1 and **Figure 2** give the situation:

- Put *angle* α on the Unit Circumference Circle (UCC) a.k.a. the *angular circle*.
- Take co-ordinates on the Unit (Radius) Circle (UR). On UR there is *arc* $\varphi = \alpha\ \Theta$.
- Find x_{ur} = xur[α] = cos[$\alpha\ \Theta$] and y_{ur} = yur[α] = sin[$\alpha\ \Theta$]. These functions thus translate the α turn into the $\{x, y\}$ co-ordinates on the unit circle.

It remains to document this further and to show that exercises become more tractable. I have considered including the paper Colignatus (2008a) in this book but the proof of concept is given in COTP (2011).

NB 1. π not only clutters traditional expressions but those expressions also implicitly use π to indicate the measurement in radians, letting students guess. NB 2. Textbooks manage to write sin(x) and cos(x) where x then both signifies the angle and the co-ordinate. NB 3. A definition sin[φ] = y / r doesn't give a function but an equation to solve.

A typical question is: Solve $\cos(\varphi)^2 - \cos(\varphi) = 0$. Solved by: $\cos(\varphi)\ (\cos(\varphi) - 1) = 0$. Thus $\cos(\varphi) = 0$ or $\cos(\varphi) = 1$. Thus $\varphi = \pi/2 + k\ \pi$ or $\varphi = 2\ \pi\ k$ rad.

This now becomes: Solve $\text{xur}[\alpha]^2 - \text{xur}[\alpha] = 0$. Solved by: $\text{xur}[\alpha]\ (\text{xur}[\alpha] - 1) = 0$. Thus $\text{xur}[\alpha] = 0$ or $\text{xur}[\alpha] = 1$. Thus $\alpha = ¼ + ½\ k$ or $\alpha = k$ UMA. Less cryptic: $\alpha = 0$, ¼ or ¾, and each subsequent full turn from there.

(Preferably though $\alpha = 4^H + k\ 2^H$ or $\alpha = k$ UMA. Less cryptic: $\alpha = 0$, 4^H or $3\ 4^H$, and each subsequent full turn from there.)

Figure 1. Angular circle ($r = \Theta^H$), unit circle (r = 1), x = Xur and y = Yur

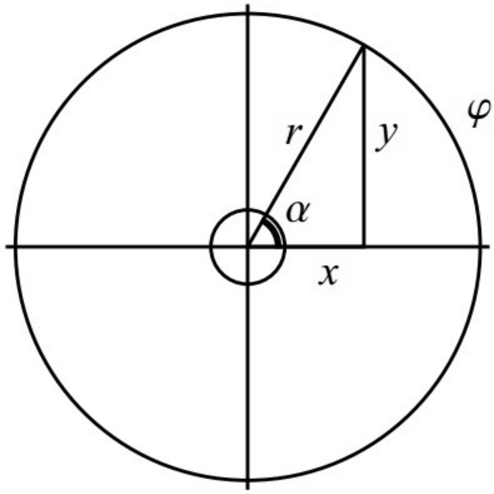
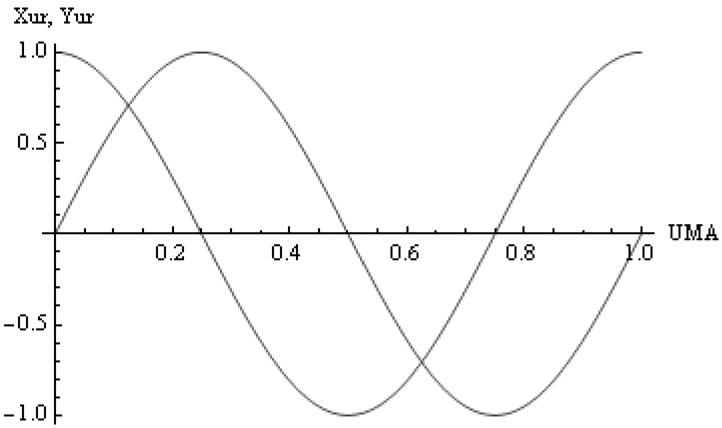


Figure 2. The functional graphs of Xur and Yur



Circles and measurement of angles (CWNN:99-100)

The Californian implementation of the *Common Core* has for Grade 4 (ages 9-10):

> "Geometric measurement: understand concepts of angle and measure angles.
>
> 5. Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:
>   a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles.
>   b. An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees.
> 6. Measure angles in whole-number degrees using a protractor. Sketch angles of specified measure.
> 7. Recognize angle measure as additive. When an angle is decomposed into non-overlapping parts, the angle measure of the whole is the sum of the angle measures of the parts. Solve addition and subtraction problems to find unknown angles on a diagram in real-world and mathematical problems, e.g., by using an equation with a symbol for the unknown angle measure." [3]

A better measure for angles is the plane itself, with unit 1.

A right angle would be 4^H = ¼ = 25% of the plane. While notation 4^H shifts understanding of fractions from division to multiplication, it may still be easier for pupils to work with integers than (such) fractions, so that 25% of the plane may be an easier measure for a right angle. Pupils might even appreciate the 250 promille measure.

The original proposal for this is in *Trig rerigged* (Colignatus (2008)). The issue translates directly to elementary school. When pupils in Grade 4 already must handle the protractor to measure angles on a scale of 360 degrees – while this is an illogical number w.r.t. the unity of the plane that they are taking sections of – then the clarity provided by *Trig rerigged* for highschool will surely be relevant for primary education too. I am at risk repeating the issue too much. *Trig rerigged* has been replaced by *Elegance with Substance* (2009, 2015) with principles, and *Conquest of the Plane* (2011) with details.

Pupils in primary education should also know that the angles of a triangle add up to half a plane. This discussion [4] is not targetted at their level but perhaps a version is feasible.

Observe the calculatory overload in the common programme. To understand an angle of 60 degrees for example, a pupil must calculate 60 / 360 to find 6^H = 1 / 6 of a circle. Thus calculation precedes understanding. The fractional form 1 / 6 invites one to continue with the calculator as well, and perhaps needlessly. To imagine what this might be, it may be transformed to 0.166... in decimal form, or 16.7% in common approximation. Instead, when the plane itself is used as the unit, then the angle 6^H stands by itself. Given the identity that 6 6^H = 1 it would be easier to see that 6^H can indeed stand by itself as "(one) per six". A transformation into decimals might not be necessary, since 16.7% is not necessarily informative. If such transformation is desired, to compare with 25%, then such a calculation cannot be avoided. Still: it is not required to do a calculation 60 / 360 to understand that 60 degrees is 6^H plane, and this would improve understanding.

[3] p32 of http://www.cde.ca.gov/be/st/ss/documents/ccssmathstandardaug2013.pdf
[4] https://boycottholland.wordpress.com/2014/06/29/euclids-fifth-postulate/

Please observe the circus: Given the Sumerian 360, there is a convention to consider special angles like 30 and 60 degrees. These have the supposedly "nice" property that sin[30] = cos[60] = ½. There is nothing particularly "nice" about this however. Don't blame the Sumerians. Blame generations of mathematicians who have been telling each other and us that this is "special". [5] But there is nothing nice or special about it. In our case we might say that sin[12^H plane] = cos[6^H plane] = 2^H, and then it is immediately clear that there is nothing special here indeed. If pupils are trained on the decimal system then it may make more sense to focus on 5%, 10% and 25% of the circle. Note that sin[5% plane] = sin[18 degree] = $(-1 + \sqrt{5})\ 4^H$. Now, isn't that *special* ? [6]

[5] http://www.themathpage.com/atrig/30-60-90-triangle.htm
[6] https://en.wikibooks.org/wiki/Trigonometry/The_sine_of_18_degrees

Insight and design

The insight is given by the following table.

	Using π and $r = \sqrt{\pi^H}$	Using Θ and $r = \Theta^H$
Circumference	$2 \pi r = ...$	$r \Theta = 1$
Surface	$\pi r^2 = 1$	$2^H r^2 \Theta = ...$

While *Trig Rerigged* 1.0 and EWS concentrated on trigonometry, and thus selected the angular circle with $r = \Theta^H$, we can also look at the circle with surface 1.

Whenever we have a unit, then this can be taken as a unit of measurement.

For example, for a measure like 25%:

- we can take this as the arc along the angular circle,
- or we can take this as the area within the circle that has a total area of 1.

The percentage remains the same.

Thus *it doesn't matter whether we look at arc or area,* as long as we know that the percentage has been taken from the appropriate circle.

The functions *Xur, Yur* and *Tur* are defined on fractions and are the same *in both cases*.

A ***circle*** is defined as the collection of points at a given distance to a center. This definition focuses on circumference. The natural constant is Θ. There is also the ***disk***, that presents the area enclosed by a circle. The natural constant for the disk is π.

The next step is the design.

(1) Probably there will be a period in primary education in which working with surfaces will be more useful. The pupils can colour the different sectors.

(2) Likely in the last year of primary education, the pupil is introduced to Θ and the angular circle. There is practice on transformations via squares, square roots and division using *H.* The Van Hiele level of understanding remains low. However, the Pythagorean theorem is already in grade 8 in the US Common Core. [7] CWNN shows Kyllian's approach for the Pythagorean Theorem for primary education. Linear and quadratic functions are also in the US Common Core. [8] It may be tested whether it is already possible to introduce *Xur, Yur* and *Tur.*

(3) In the first phase of secondary education, there will be informal deduction of the properties of the circle, with area and measurement of angles, with *Xur, Yur* and *Tur.*

(4) Definitely there is a moment in secondary education when arcs are selected as the best representation for trigonometry. We want to establish that the derivatives of sine and cosine are given by each other, which necessitates radians.

The following gives an outline for the bottom-up presentation.

[7] http://www.corestandards.org/Math/Content/8/G/
[8] http://www.corestandards.org/Math/Content/8/F/

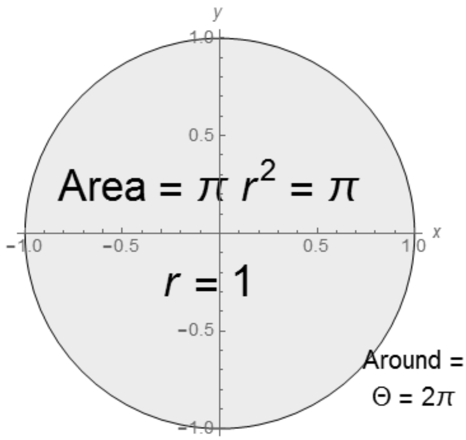
Introduction of angles in primary education

When pupils in primary education are introduced to the circle and measurement of angles, then they can start with, and also see Figure 3:

- The circle with radius 1 has a disk with area $\pi \approx 3.14....$
- There is a disk with area 1. Let us call this the *hook disk*. Angles can be measured as hooks, that are fractional sectors, or $0 \leq h \leq 1$. This circle has radius $r \approx 0.564...$

Thus the pupil can measure angles in terms of hooks, which are fractional sectors of the hook disk. An example of a hook of 12.5% is given in Figure 4.

Figure 3. Area of the unit circle is π. Area of the hook disk is 1



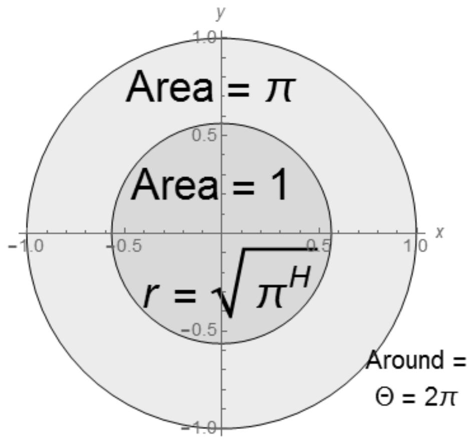
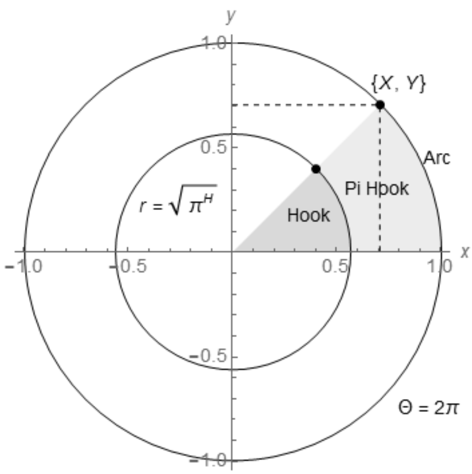


Figure 4. A hook of 12.5%



- It is only later when the formula for the area is introduced, or squaring in general, or the use of the square root. However, these expressions are included here for convenience.
- CWNN mentions that primary school likely allows the introduction of co-ordinates and vectors (Van Hiele). Thus we could introduce the specific coördinates {X, Y} on the unit circle, that have the property $X^2 + Y^2 = 1$. Pupils can practice on {X, Y} = {x, y} r^H.
- CWNN introduces $H = -1$ for negative numbers. Use the circle with i, $H = i^2$ and i^3.

Hook disk

Potentially, the pupil starts with the hook disk. It will be useful to establish the addition of hooks, see the US Common Core requirement above.

The addition of hooks might seem fairly obvious here. But it may be shown in the classroom with a radius of 3 meters what the meaning of the addition of hooks is: a small step for some is a long walk for others.

At one stage the pupil shifts from the hook disk to looking at the unit circle. The obvious question is how the larger sector must be called, that starts with the hook and continues to the unit circle. A suggestion is to call this the "Pi hook".

- All hooks multiplied by π generate the appropriate sector for the unit circle.
- For example, the hook disk has area 1, and multiplied by π, it generates the area of the unit circle.
- There is a proportionality of sector areas in concentric circles, determined by the areas of the whole circles.
- There is a proportionality of arcs in concentric circles, determined by the radiuses.
- The pupils can discover that they must be cautious on this. The area goes by the square or the radius. Later in education, students can prove this.

  (It shows in the formulas. Perhaps: prove for circles with r and $R = a\, r$ that the areas are proportional to the square of the ratio of the radiuses: $\pi\, (a\, r)^2\, (\pi\, r^2)^H = a^2$.)

An issue of design

For design there are two options and the one presented above is my preference. It still is useful to mention the other option. This can best be done in a table.

	Presented above	*Rejected*
Name of disk with area 1	Hook	Angle, like on angular circle
Name of sector of unit circle	Pi hook	Hook
Advantage	Dimensions always matter. Distinguish pure number from dimensional context.	The dimension of numbers in [0, 1] would not seem to matter.
Disadvantage	More explanation about angle, hook and Pi hook	Confusion of angle and hook. Need for hook / Pi. There is a useless factor 2 between radian and hook.

Towards the end of primary education

Towards the end of primary education, the pupil will be introduced to the angular circle.

- The angular circle has circumference 1.
- The angle concerns an arc along the angular circle.

Figure 5 shows hook disk and angular circle separately, Figure 6 combines the circles. The hook value of sector OCD gives a fraction of area 1. The angle or arc AB gives a fraction of circumference 1. The fractional values are the same.

Figure 5. Hook disk and angular circle

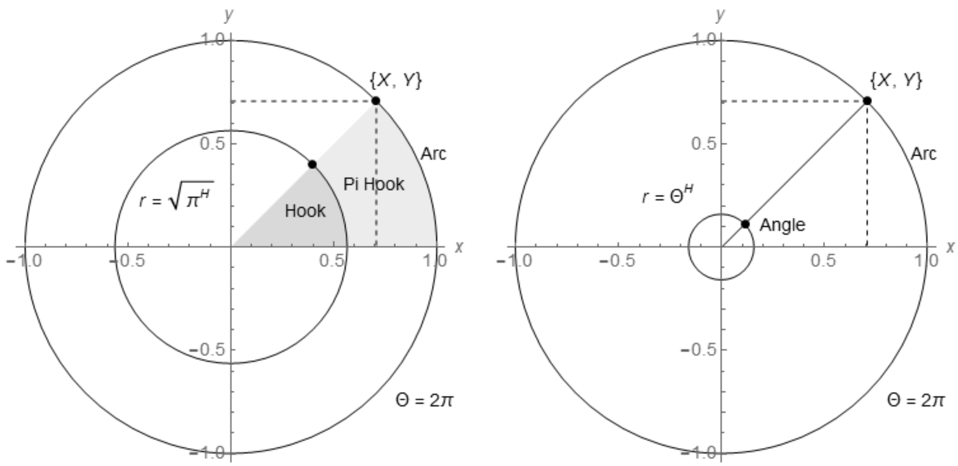
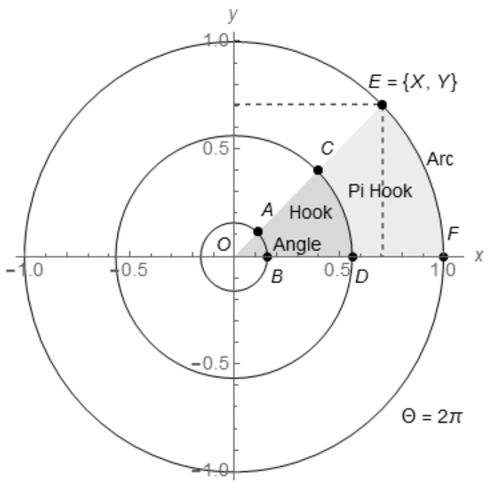


Figure 6. The circles combined



Xur and Yur in primary education ?

The US Common Core mentions functions in grade 8. [9] These are the linear and quadratic function now. The US Common Core has the sine and cosine only in highschool. [10]

For primary school:

- It is conceivable to introduce the functions *Xur, Yur* and *Tur*. This can be done first for the hook, as fraction of the hook disk.
- Subsequently, there can be discussion on the angle, as fraction of the angular circle.

Hook function

Figure 7 displays the hook as function of *X.* Note the didactic improvements:

(1) For some readers it might come as a surprise that *h* = ArcCos[*X*] / (2 π). They might never have realised that arccos gives a relevant area. It goes to show how undidactic education currently is.

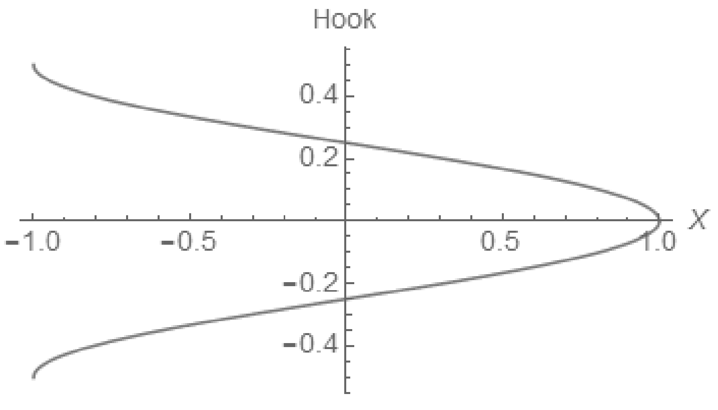
(2) It is better to start with *h* = *hook*[*X*] and later derive *X* = *Xur*[*h*].

  (a) The values of {*X, Y*} can be found as the co-ordinates, and the size of the hook (sector of the disk of size 1) depends upon these co-ordinates.

  (b) Traditionally, when Cos[φ] = (adjacent / hypothenuse) is introduced, then it actually is an equation that must be solved for φ. Cos[φ] does not have the form of say *f*[*x*] = *x*^2, with an expression that generates a result. It is better to introduce φ = ArcCos[adjacent / hypothenuse].

(3) A value of *X* associates with two hooks. The hook function necessitates the distinction between functions and correspondences. When functions are presented in primary school, it is useful to have this distinction there too.

Figure 7. Hook function



[9] http://www.corestandards.org/Math/Content/8/F/
[10] http://www.corestandards.org/Math/Content/HSF/TF/

Domain and range

Wolfram (1996), *Mathematica,* allows functions to have the same name but different domains and ranges. For example:

f[x_Integer] := x^2

f[x_String] := x <> " times " <> x   (*concatenation*)

In school math one can define such functions but they must still have different names, like "*f* for integers" and "*f* for strings".

In EWS, Xur and Yur have been defined on angles, i.e. arcs along the angular circle (with circumference 1). When this definition is seen as being neutral on real numbers, then the functions can also be used on hooks, i.e. areas on the hook disk (with area 1). However, when one thinks of it in terms of Xur[a_Angle] and Xur[h_Hook] then it may be that one makes the issue more complicated than needed with "Xur for angles" and "Xur for hooks". Obviously, the contexts for the real numbers differ, and pupils must grow aware of this.

Graphs and dynamic displays

The graphs have been shown in Figure 2.

Dynamic displays for primary school have higher requirements for comprehensibility. Dynamic displays of sine and cosine are available at the Wolfram Demonstrations project. There is also one of Palais and Meier.

- I would tend to start with Patson and Cuthberth (undated), since this allows to manipulate the point on the circle directly.
- Subsequently, Schulz (undated) better displays that the arc is on the horizontal axis.
- Palais and Meier (undated) have a fine display with a rolling circle. There is a less clear link between the length of the arc and the distance rolled: using the same colour would help. A dot that traces the sine curve would help too. In that manner the relation to the y co-ordinate will also be clearer. It is nice to flip the axes for the cosine. A problem remains that the axes for arc and x and y overlap, and those better be separated, as in the Demonstrations project.
- The animation at wikipedia for the sine is fairly good, but one would want to be able to manipulate the position, and the choice of yellow for the vertical position is too light. [11]

This indicates that fine displays still require some attention. Overall, it seems possible to design a dynamic display for *Xur, Yur* and *Tur* that pupils in elementary school would be likely to understand. If not used in elementary school, the display will still be useful for the first phase in secondary education.

Calculations

The obvious question is also why pupils in primary education would use these functions.

I suppose that it would be useful to know also at this age that an angle can be computed from measuring the sides of a right-angled triangle. Using Angle ~ Hook ~ ArcXur:

x = hypothenuse *Xur*[angle]   ↔   angle = *ArcXur*[x / hypothenuse]

[11] https://en.wikipedia.org/wiki/Trigonometry#/media/File:Sine_curve_drawing_animation.gif

One would explain that "hypothenuse" is the name of the radius, when the circle has not been explicitly drawn. But it might be good exercise to draw the circle anyhow for a number of times.

Though it is an option to see whether this can be done in primary school, it may be mentioned that I don't tend to be much in favour of much of trigonometry in secondary school anyway. When relations can be reduced to powers of *e,* then we should consider trying to use that approach.

Highschool

COTP (2011) already contains a *proof of concept* how *Trig Rerigged* 1 can be designed for highschool.

This present *Trig Rerigged* 2.0 provides an amendment to start in elementary school with hooks, as fractions of the hook disk with area 1.

It is a consideration that pupils first learn about hooks as areas and then must grow aware of angles as arcs on the angular circle, and then that students in highschool must learn to handle both angles on the angular circle and radians on the unit circle. One might argue that unlearning old ways takes more effort. Why not have radians from the start ?

The answer lies in empirical research. It are the students themselves who must show what works best for them. The next real question is how to specify and set up such empirical research.

Above design is only a suggestion. It is based on these ideas:

- hooks, angles and radians are not in conflict, so there is no need of unlearning
- having a rich environment of notions helps to cement the whole
- having only radians (or traditional 360 degrees, but the same approach) is not rich, and you get it or you don't - and if you don't get it then you must repeat it, which is awkward since you didn't get it in the first place
- starting with surfaces will work better for younger pupils
- using units of account with value 1 is preferable
- ending with radians is required because of the derivatives
- it is important that pupils develop these fundamental insights. This is where the threshold or bottleneck lies. When the fundamental notions of angles and xur and yur are not grasped as natural concepts, then also radians, sine and cosine are not grasped as more involved concepts, and then doing more complex math becomes a problem. If instead this insight in the natural concepts arises, then other concepts present less of an issue. Formulas for conic sections and spheres can be looked up in a table if required, and copied into a computer algebra package (e.g. *Mathematica*).

PM 1. David Butler's suggestion of $\eta = 4^H \Theta$ might be a useful symbol on occasion. [12]

PM 2. The notion of "rational trigonometry" by Norman Wildberger is useful to know about, since it answers a lingering question whether reals could be avoided (for engineering). Wildberger shows that "spreads" are not proportional to angles, and thus I deem them less useful for education. Real numbers are an important result of mathematics for science, and there is no reason to avoid them. [13] I have not read his book "Divine Proportions" and thus these remarks are preliminary. COTP tries to advance as fast as possible to vectors and linear algebra too, and results of "rational trigonomety" might generate useful applications.

[12] https://www.youtube.com/watch?v=1qpVdwizdvI
[13] https://en.wikipedia.org/wiki/Rational_trigonometry and http://web.maths.unsw.edu.au/~norman/Rational1.htm

Acknowledgement and protest

The development of these ideas has benefitted from e.g. Palais (2001), Hartl (2010), Cavers (2011). I am grateful for this input. I discovered Cavers (2011) only at the end of August 2016. Section 3.1 was helpful for me to remember again that *Trig Rerigged* concerned trigonometry and not the calculation of area. I had already made the point in Colignatus (2014a) too but it was useful to look at it afresh in 2016. From there it was a short step to the insight on page 15 and the graphs of page 16 above.

The overall best overview page is given by Harremoës (2012), who also provides a portal to the various authors. Recommended is design of a "centiturn protractor", for which one would wish a 3D printer code. [14]

Some drawbacks on the approaches by Palais, Hartl and Cavers must be mentioned too.

- The issue on π versus 2π has gotten the "framing" as if this would concern a fundamental mathematical question, which has to be decided upon by mathematical criteria. Check the "manifestos" by Hartl (2010) and Cavers (2011). I don't think that this is a mathematically relevant issue, but I am no mathematician but an econometrician and qualified teacher of mathematics.
- As remarked on page 10: There is the tau-community who propose to use tau or $\tau = \Theta = 2\pi$. Earlier I considered this option too but rejected it because of the similarity with *r,* the symbol for the radius. This similarity will cause much confusion, especially in handwriting in exams.
- These authors mention education but not as their main concern. They appear not to be open to the argument for trigonometry in general with function Xur and Yur, as formulated in EWS (2009, 2015), COTP (2011), and my additional argumentation on my weblog.
- The tau-community draws attention from readers, including from journalists who reach more readers, but their issue on tau is mathematically irrelevant and didactically unwise, and thus reminds of the "math wars" [15] between mathematicians motivated by abstract thought, that poisons the discussion on education, and that drives away readers who would be interested in a real issue. My potential readership will be quite wary when I would try to reach them on *Trig Rerigged,* EWS and COTP.

Palais states on his website:

> "I was wondering why students didn't realize $\pi/2$ was a quarter of the way around the unit circle, so that $(\cos(\pi/2), \sin(\pi/2)) = (0,1)$, and suddenly the disconnect between the half and the quarter forced me to realize that it wasn't their fault, it was ours. π was wrong!" [16]

However, his subsequent step is to develop the case for 2π and not look further into the understanding of students.

For me, I looked at the understanding of students, and when *Trig Rerigged* looked at trigonometry I thus arrived at the functions *Xur* and *Yur.* Archi was only part of the package. I agree that my focus was on trigonometry and not on measurement of area. Thus the argument of Cavers (2011) section 3.1 eluded me till I read it in August 2016.

[14] http://www.harremoes.dk/Peter/centitunprotractor.pdf
[15] https://en.wikipedia.org/wiki/Math_wars
[16] http://www.math.utah.edu/~palais/pi.html

COTP:175 mentions the option to normalise to a surface of 1 but sees "no particular reason yet". The insight on page 15 above is rather convincing however.

Let me restate that I tend to allow for both symbols. Both represent the same constant, but with a factor that depends upon application.

- A ***circle*** is defined as the collection of points at a given distance to a center. This definition focuses on circumference. The natural constant is Θ.
- There is also the ***disk***, that presents the area enclosed by a circle. The natural constant for the disk is π.
- It is a useful exercise for student to show that $\Theta = 2\pi$.

Unfortunately, In Holland there was science journalist George van Hal (2011, 2016) and the Redactie Euclides (2016) who referred to professor Jan Hogendijk at the University of Utrecht:

> "According to me the one constant for the circle is no better than the other one. For me, fighting against Pi is like the tilting at windmills by Don Quixote." [17] (my translation)

They also quote professor Frits Beukers at Utrecht:

> "According to me it is typical of these times that people nowadays, after centuries of use, make a fuss about such silly issues. Probably the easy access of internet contributes to this." (my translation)

Unfortunately, this again neglects the didactics of mathematics. Who will hear or read about my work will become wary as if this were a vain issue. It is remarkable that the editors of Euclides (2016) referred to Van Hal (2016) and reprinted these quotes uncritically, even though Euclides is the journal of the Dutch association of teachers of mathematics. It would be very useful when these pronouncement would be corrected: that they do not apply to the didactics of mathematics. Hence, see Colignatus (2016b) in Dutch for a critical report and protest about this situation.

[17] https://en.wikipedia.org/wiki/Tilting_at_windmills

Conclusions

Above prospect seems promising at least for highschool.

I am not qualified to judge for primary education but the suggestions might help.

Literature

PM 1. Colignatus is the name of Thomas Cool in science. See http://thomascool.eu.
PM 2. References in footnotes might not be repeated here.

Cavers,M. (2011), "The Pi Manifesto", July 4, http://www.thepimanifesto.com

Colignatus (1981 unpublished, 2007, 2011), "A Logic of Exceptions" (ALOE), http://thomascool.eu/Papers/ALOE/Index.html

Colignatus (2008), "Trig rerigged", **Legacy:** see EWS for principles and COTP for details

Colignatus (2009, 2015), “Elegance with Substance”, 1st edition Dutch University-Press, 2nd edition mijnbestseller.nl, http://thomascool.eu/Papers/Math/Index.html

Colignatus (2011), “Conquest of the Plane”, http://thomascool.eu/Papers/COTP/Index.html

Colignatus (2012a), “Een kind wil aardige en geen gemene getallen” (KWAG), http://thomascool.eu/Papers/AardigeGetallen/Index.html

Colignatus (2012b), "A child wants nice and not mean numbers", **Legacy** collection, parts of KWAG in English, no longer on the web (and "not" instead of "no")

Colignatus (2012c), "Pas je uitleg aan. Interview met Yvonne Killian", Euclides 88 (2) p 83-84

Colignatus (2014a), "Cirkels in vierkanten. Oppervlakte leidt tot π en omtrek leidt tot Θ = 2 π (archi = 2 pi)", verkennend en niet zo sterk opstel over archi o.g.v. verhoudingen, januari en april 2014, http://thomascool.eu/Papers/AardigeGetallen/2014-01-09-Cirkels-in-vierkanten.pdf

Colignatus (2014b), "Pierre van Hiele and David Tall: Getting the facts right", http://arxiv.org/abs/1408.1930

Colignatus (2015), "A child wants nice and no mean numbers", http://thomascool.eu/Papers/NiceNumbers/Index.html

Colignatus (2016a), "Negatieve getallen, breuken en Simon Stevin Instituut", http://thomascool.eu/Papers/AardigeGetallen/2016-01-10-negatieve-getallen-breuken-Simon-Stevin-Instituut.pdf

Colignatus (2016b), "Stop het schelden en lasteren met Don Quichot en die windmolens". http://thomascool.eu/Papers/AardigeGetallen/2016-07-04-Nav-JanHogendijk-en-DonQuichot.pdf

Colignatus (2016c), "Trigonometry Rerigged 2.0", https://boycottholland.wordpress.com/2016/09/05/trigonometry-rerigged-2-0/

Hal, G. van (2011), "Opstand tegen π", website 5 maart – "Dit verhaal is eerder verschenen in NWT Magazine" - http://www.georgevanhal.nl/artikelen/opstand-tegen-%CF%80

Hal, G. van (2016), "π-dag vieren? Nee, het is tijd voor opstand tegen π!", NewScientist.nl, 14 maart, http://www.newscientist.nl/nieuws/opstand-tegen-%CF%80

Harremoës, P. (2012), "Al-Kashi’s constant τ", http://www.harremoes.dk/Peter/Undervis/Turnpage/Turnpage1.html

Hartl, M. (2010), "The Tau Manifesto", http://tauday.com

Hiele, P.M. van (1973), "Begrip en inzicht", Muusses

Killian, Y. (2006), "De stelling van Pythagoras voor de brugklas of groep 8", Euclides 81 (7), p 338

Killian, Y. (2012), "De stelling van Pythagoras voor de brugklas of groep 8. Deel 2. Praktijkervaring en een bijbehorend bewjs", Euclides 88 (5), p 250-251 (plus erratum)

Limpens, G. (2010), "Boekbespreking: "Elegance with Substance"", Euclides 86-3, p130-131, http://thomascool.eu/Papers/Math/2010-12-Euclides-86-3-p130-131-a.jpg

Palais, R. (2001), "π is wrong!", The Mathematical Intelligencer, Springer-Verlag New York Volume 23, Number 3, 2001, pp. 7-8 http://www.math.utah.edu/~palais/pi.html

Palais, R. and D. Meier (undated), "The natural sine curve", http://www.math.utah.edu/~palais/cossin.html and http://www.math.utah.edu/~palais/sine.html

Patson, N. and A. Cuthberth (undated), "Relationship of Sine and Cosine to the Unit Circle" from the Wolfram Demonstrations Project http://demonstrations.wolfram.com/RelationshipOfSineAndCosineToTheUnitCircle/

Redactie Euclides (2016), "Aanval op het getal ", Euclides 91-7, juni 2016, pagina 7

Schulz, E. (undated), "Sine, Cosine, Tangent and the Unit Circle" from the Wolfram Demonstrations Project, http://demonstrations.wolfram.com/SineCosineTangentAndTheUnitCircle/

Wolfram, S. (1996), "The *Mathematica* Book", 3rd edition, Wolfram Media / CUP

Index

A

Addition, 13
Algebra, 4
ALOE, A Logic of Exceptions (book), 26
Angle, 10, 11, 13
Angular circle, see UCC, 11, 12
Approximation, 10, 13
Archi (symbol Θ = 2π), 10, 11, 15, 23
Archimede, 10

C

Calculator, 10, 13
Calculus, 10
Cause, 10, 11, 23
Circle, 10, 11, 12, 13, 14
Common Core, 13
Communicate, -ion, 11
Cool, Th., 4, 26
Co-ordinates, 11
Cosine, cos, 11, 12, 14
COTP, Conquest of the Plane, 11
COTP, Conquest of the Plane (book), 26

D

Decimal (system), 13, 14
Degree, 14
Derivative, differentiation, 11
Diameter, 10, 11
Didactic, -ics, -cal, 4
Dimensionless, 11
Division, 13

E

Econometrics, 4
Education, 4, 13
EWS, Elegance with Substance (book), 5, 26

F

Fraction, 13
Function, 11

G

Geometry, 4
Greece, -k, 10
Groningen, 4

H

Hiele, P.M. van, - levels of abstraction, 26

I

Integral, 10

K

Killian, Y., 26
KWAG (book), 26

L

Leiden, 4

M

Measure, -ment, -ing, 10, 11, 13
Meter, 11
Methodology, 4
Multiply, -ication, 13

N

Notation, 13
Number theory, 4

P

Palais, R., 10
Political Economy, 4
Program, programme, 13
Property, 14
Pythagoras, 26

R

Radian, 10, 11
Ratio, 10, 11

S

Scale (nominal, ordinal, interval, ratio), 13
School, elementary (primary education), 3, 13
School, highschool, 13
Sine, sin, 11, 14
Sumer, 11
Sumer, -ian, 14
Surface, 10

T

Tall, D.O., 26
Tangent, 11
Tau (see Archi), 10, 23
Teacher, teaching, 4
Theta, 10
Trigonometry, 10, 11
Turn (angle measure), 9, 10, 11, 12

U

UCC, Unit Circumference Circle, angular circle, 11
UMA, 11, 12
Unit (radius) circle, 11, 12

V

Vector, 4

W

Wiskunde, 4

X

Xur, 11, 12

Y

Yur, 11, 12

Π

π, 10, 11, 12, 23