Pierre van Hiele and David Tall: Getting the facts right

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http://thomascool.eu
July 27-28, August 29 2014

arXiv math.HO (History and Overview)
MSC2010 codes 97A30, 97C70, 97B10 (research in mathematics education)

Abstract

Pierre van Hiele (1909-2010) claimed wide application for the Van Hiele levels in understanding, both for more disciplines and for different subjects in mathematics. David Tall (2013) suggests that Van Hiele only saw application to geometry. Tall claims that he himself now extends to wider application. Getting the facts right, it can be observed that Tall misread Van Hiele (2002). It remains important that Tall supports the wide application of Van Hiele's theory. There appears to exist a general lack of understanding of the Van Hiele - Freudenthal combination since 1957. Part of this explains the current situation in the education in mathematics and the research on this.

Introduction

Pierre van Hiele (1909-2010) and Dina (Dieke) Van Hiele-Geldof (1911-1958) were inspired by Piaget's idea of levels of understanding of mathematics, notably linked to age. They tested that idea, and defined and empirically developed the Van Hiele 'levels of abstraction in understanding mathematics', more independent of age, in their two separate theses in 1957. As Piaget presented a general theory, the Van Hieles presented an alternative general theory. The usual reference is to Van Hiele (1986) in English, though I am limited to Van Hiele (1973) in Dutch.

In the following we will regard mathematics as equivalent if not identical to abstraction itself. Thus the Van Hiele theory concerns the general didactics of abstraction and abstract concepts. Abstraction is often seen as 'higher' and possibly more complex, but it may also be seen as the elimination of selected aspects in concrete elements, so that it may also be regarded as rather simple (so that we can also understand that the human brain is capable of abstract thought, as it isn't such a particularly complex thing to do).

Below discussion will not recall what the 'Van Hiele theory of levels' actually is. This is a deliberate choice. The main objective of this paper is to alert the reader to the wealth of insight in didactics in the work by the Van Hieles, as applicable to didactics in general, and thus it is consistent to refer only. There are more expositions available in English, see Van Hiele (1986) or **Appendix A** on Van Hiele (1959) at ERIC.

David Tall (2013), *"How humans learn to think mathematically. Exploring the three worlds of mathematics",* suggests that Pierre van Hiele had a limited understanding of the portent of this theory, notably that Van Hiele saw it limited to geometry and not applicable for e.g. algebra or even other disciplines. Tall also suggests that he himself extends our understanding to the wider portent of those levels. However, it appears that Tall's suggestions are based upon a misreading of Van Hiele (2002). The truth is that Van Hiele was quite aware of the fundamental nature of his and Dieke's result. It is better to get the facts right and indeed alert students of education (not only mathematics) to the wealth that can be found in Van Hiele's work.

There is indeed some tendency, like on wikipedia 2014, to restrict the Van Hiele levels to geometry alone but it wouldn't be right and useful if this became a general misunderstanding. By analogy, when someone mentions the number 4 as an example of an even number, it is invalid to infer that this person thinks that 4 gives all even numbers. Van Hiele had been teaching mathematics on various topics for decades and had observed the levels in various topics, and indeed in various disciplines, like chemistry and didactics itself. The thesis merely took geometry as an example, or, rather as the example *par excellence*, as geometry is a foundation stone for mathematics as the art and science of demonstration. His thesis provided in mathematical fashion both a *definition* of the levels in human understanding and an *existence proof*. See **Appendix A** also on the proper translation of the title of the thesis.

The impact of misconception can be large. The task group on learning processes of the 2008 US National Mathematics Advisory Panel states, see Geary et al. (2008:4:xxi-xxii): "The van Hiele model (1986) has been the dominant theory of geometric reasoning in mathematics education for the past several decades. (...) Research shows that the van Hiele theory provides a generally valid description of the development of students' geometric reasoning, yet this area of research is still in its infancy." Apparently the levels are not applied to other subjects either.

Professor Tall's personal reappraisal of Van Hiele's work is important for the wider recognition of that work. The convincing part in Tall's argumentation for the wider application is basically no different from the argumentation that Van Hiele already provided, which is another corroboration of the original insight. A nuance is that research in the education of mathematics has provided additional corroboration since 1957 also by Van Hiele himself. Another nuance is that Tall (2013) unfortunately still has a limited understanding of the Van Hiele levels and introduces some misunderstandings. While this personal reappraisal is important, there might be the danger that researchers and students now would focus on Tall (2013) as the most recent text while it would be advisable to study the original work by the Van Hieles.

The text below has also some comments on the role of Hans Freudenthal (1905-1990) with respect to Van Hiele's work. Given that both Van Hiele and Freudenthal worked in Holland, and Pierre wrote his thesis under Freudenthal (with second advisor Langeveld), one may wonder how their approaches compare. Outsiders like David Tall might be able to observe at a larger distance, and see broader strokes. It appears however that professor Tall at least misses some information that is available in Dutch. This is rather a general situation.

The article has the following structure. Below we first focus on Tall (2013) and his supposed evidence in Van Hiele (2002), and give the quotes that highlight both the claim and the misunderstanding. Subsequently, we provide quotes from Dutch sources now in English translation. Subsequently we compare the Van Hiele levels with Tall's diagram of such levels, to check that we are speaking about the same things. Some of Tall's misunderstandings generate a somewhat distorted model, which causes difficult semantics whether we are really speaking about the same things: but overall this sameness could be accepted when the misunderstandings are recognized for what they are. Subsequently, we consider the triad Tall, Van Hiele & Freudenthal. An enlightening observation is that also Freudenthal claimed a general theory derived from Van Hiele, and also provided distorted information as well, and that this seems to have affected Tall's perception as well. Appendix B contains supporting translations from a Dutch thesis by La Bastide-Van Gemert (2006) of which Chapter 7 looked at Freudenthal on the Van Hiele levels. Perhaps this article might be re-organised into a focus on the three authors in general. However, Tall (2013) is the most recent publication and it is from a world-renowned researcher who after retirement takes stock of his lifetime work. It is useful to basically focus on "Tall about Van Hiele" as we do now.

Quoting the two authors

Tall (2013:153): "As we consider the whole framework of development of mathematical thinking, we see a prescient meaning in the title of van Hiele's book *Structure and Insight.* [footnote referring to Van Hiele (1986)] Even though he saw his theoretical development of levels of structure applying only to geometry and not algebra [footnote referring to Van Hiele (2002)], his broad development, interpreted as structural abstraction through *recognition, description, definition* and *deduction,* can now be extended to apply throught the whole of mathematics." (Note the limitation to "structural" abstraction. This comes back below.)

Tall (2013:430): "It was only in 2011, when Pierre van Hiele passed away at the grand old age of 100, that I explicitly realized something that I had 'known' all along: that the structural abstraction through *recognition, description, definition* and *deduction* applied successively to the three worlds as concepts in geometry, arithmetic and algebra, and formal mathematics were recognized, described, defined and deduced using appropriate forms of proof."

When I queried professor Tall on this, he sent me a copy of Van Hiele (2002) so that it has been verified that he regards that article as the "proof" that Van Hiele would have only a limited understanding of the portent of his theory. It appears to be a misunderstanding, and I don't think that such "proof" could be found elsewhere either. We can constrast above two quotes from Tall (2013) with Van Hiele (2002).

Van Hiele (2002:46) in his conclusion, with the word "disciplines" referring to also physics, chemistry, biology, economics, medicine, language studies, and so on:

> “In most disciplines there are different levels of thinking: the visual level, the descriptive level and the theoretical level.”

Indeed, Van Hiele gives various examples in my copy of “Begrip en Inzicht” (1973), which I presume will be more extended in English in “Structure and Insight” (1986) that however is not in my possession.

Thus, Van Hiele was aware of the wide portent of the theory of levels of abstraction.

How could it be, that Tall did come to think otherwise ? One aspect will be an issue of reading well. One aspect might be a general misconception that the theory applied only to geometry. Readers who suffer this misconception might no longer read carefully. It may be observed that geometry has somewhat been reduced in the education in mathematics so that, if there is a misconception that the Van Hiele levels apply only to geometry, then this is one avenue to explain the reduced attention for the Van Hiele levels of abstraction.

The following quotes are relevant for the view on algebra. Van Hiele (2002) does not give a technical developed of particular levels for arithmetic and algebra, but the point is that it shows that Van Hiele thought about the teaching of those subject matters in terms of levels. It is a wrong reading by Tall not to recognise this.

Van Hiele (2002:28) warns: "The problems in algebra that cause instrumental thinking have nothing to do with level elevation since the Van Hiele levels do not apply to that part of algebra. People applied terms such as 'abstraction' and 'reflection' to the stages leading from one level to the next. This resulted in a confusion of tongues: we were talking about completely different things." Thus "part of algebra" should not be mistaken for all algebra.

Van Hiele (2002:39): "The transition from arithmetic to algebra can not be considered the transition to a new level. Letters can be used to indicate variables, but with variables children are acquainted already. Letters can be used to indicate an unknown quantity, but this too is not new." Indeed, Tall (2013:105) confirms: "It is well known that students have much greater success in solving an equation with *x* only on one side. [footnote] Filloy and Rojano [footnote] named this phenomenon the 'didactic cut' between arithmetic and algebra." Thus note:

- Van Hiele uses "algebra" to mean that the use of a single variable would still be classified as "algebra" even though it isn't really different from arithmetic.
- Filloy and Rojana use "algebra" as distinctive from arithmetic, so that a single occurrence of an unknown would not be classified as "algebra". (Or they use a dual sense.)
- Apart from this issue in terminology, the diagnosis is the same.
- And this isn't evidence that Van Hiele had a limited view on the portent of levels, but rather the opposite.

Van Hiele (2002:43): "The examples Skemp mentions in his article about I2, R2 and L2 do not have any relations with a level transition. They are part of algebra in which topic, as I have emphasised before, normally level transitions do not occur." Again "part of algebra" should not be read as all algebra. In this case we must observe that the sentence can be bracketed in different ways: "They are (part of algebra) in which topic ..." or as "They are part of (algebra in which topic ....)". An observant reader will be aware of this issue and rely on the rest of the text to determine the proper bracketing. My diagnosis is that professor Tall focussed on this sentence and mislaid the "which". However, the other sentences and in particular the conclusion in the very same article on page 46 (Van Hiele (2002:46), quoted above) make clear that this shouldn't be done. I have asked professor Tall whether this particular misunderstanding was the source of his suggestions indeed, but haven't received an answer on this particular question yet.

Thus, Van Hiele was aware of the portent of his theory, contrary to what Tall states.

On might hold that it doesn't matter whether Tall read something wrong and that the useful issue is to arrive at a working theory for didactics. However, the point is that Van Hiele already presented such a theory. The derived issue is quite limited here: that Tall creates a confusion and that it helps to eliminate the reasons upon which he based that confusion: (i) so that others do not follow that same road, (ii) so that readers of Tall (2013) are alerted to this confusion there. Overall, students of didactics are advised to consider the original Van Hiele texts.

Some other sources

It will be useful to point some other (Dutch) sources and publications by Van Hiele also explicitly on arithmetic and algebra.

The opening paragraph in the Introduction to his thesis Van Hiele (1957:vii):

> "Insight" is a concept that can present itself to us in different fashions. The meaning of the different aspects differs, depending upon the context for which one studies insight. In the following study here I have occupied myself with the position, that insight takes in the context of didactics and even more special in the didactics of geometry. This limitation causes that the conclusions that are arrived at cannot be regarded without additional research as "generally valid". From this study it may however appear that there shouldn't be expected differences in principle between "insight in geometry" and "insight in mathematics in general". I am also under the impression that "insight in mathematics" will be quite similar in many respects with "insight in non-mathematical school subjects". It might however be that insight plays a much less fundamental role in some school subjects other than mathematics, without the implication that such school subjects would have to be less important for the child." [1]

Van Hiele (1957) is remarkably consistent in focussing on geometry, but discusses in passing that there are also levels of insight in algebra (p131-132) and in understanding didactics and insight itself (p201-204), while he also discusses Langeveld's study of

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[1] My translation of: ""Inzicht" is een begrip, dat zich op verschillende wijzen aan ons kan voordoen. De betekenis van de verschillende aspekten varieert, al naar gelang men het inzicht in de ene of in de andere samenhang bestudeert. In de hier volgende studie heb ik mij speciaal bezig gehouden met de plaats, die het inzicht inneemt in de didaktische kontekst en nog meer speciaal in de didaktiek van de meetkunde. Deze beperking maakt, dat men de gevonden konkusies niet zonder nader onderzoek als "algemeen geldig" mag beschouwen. Uit deze studie moge echter blijken, dat er in ieder geval tussen "inzicht in meetkunde" en "inzicht in wiskunde in het algemeen" geen principiële verschillen verwacht mogen worden. Ook schijnt het mij toe, dat "inzicht in wiskunde" toch nog op vele punten overeenkomst zal vertonen met "inzicht in niet-wiskundige schoolvakken". Het zou echter wel kunnen zijn, dat in sommige schoolvakken het inzicht een minder fundamentele rol vervult dan in de wiskunde, zonder dat daardoor deze schoolvakken voor het kind minder belangrijk behoeven te zijn."

learning checkers (p105). Overall, one cannot conclude that Van Hiele thought that his new theory of levels of insight was limited to geometry only.

La Bastide-Van Gemert (2006) - henceforth LB-VG - reports about Van Hiele on writing his thesis in 1957: "He restricted himself to the education in geometry, since he did not see differences in principle between insight in geometry and insight in mathematics in general." [2] She also reports that the two Van Hieles wrote in an article in *Euclides*, the Dutch journal for teachers of mathematics, in 1957: "The here for geometry given approach can namely also be used for other disciplines." [3] They include some conditions that are not relevant here.

Van Hiele (1959) considers thought and then focuses on geometry as an example. His opening line is: "The art of teaching is a meeting of three elements: teacher, student, and subject matter." He speaks about mathematics in general rather than geometry.

Geometry enters only when: "The following example will illustrate what I mean." - with that example taken from geometry, all aware that the subject matter might affect the analysis. However he continues speaking about "optimal mathematical training" in general. A useful quote is: "In general, the teacher and the student speak a very different language. We can express this by saying: they think on different levels. Analysis of geometry indicates about five different levels."

It may be useful to know that Van Hiele's attention for the role of language derived from the teachings by Gerrit Mannoury (1867-1956), professor in mathematics at the university of Amsterdam in 1917-1937 who did much research in semiotics (or in Dutch: significa). Mannoury explained already quite early what Ludwig Wittgenstein rediscovered and rephrased more succinctly: the meaning of a word is its usage (i.e. the notion of *language games*). Van Hiele (1959) gives the closing statement: not about geometry only but for each discipline:

> "The heart of the idea of levels of thought lies in the statement that in each scientific discipline, it is possible to think and to reason at different levels, and that this reasoning calls for different languages. These languages sometimes use the same linguistic symbols, but these symbols do not have the same meaning in such a case, and are connected in a different way to other linguistic symbols. This situation is an obstacle to the exchange of views which goes on between teacher and student about the subject matter being taught. It can perhaps be considered the fundamental problem of didactics."

Van Hiele (1962) *"The relation between theory and problems in arithmetic and algebra"* is a chapter in an ICMI report by Freudenthal (ed) 1962. I haven't been able to check yet whether the levels are applied but would be amazed if they would not appear.

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[2] My translation of: "Hij beperkte zich daarbij tot het meetkundeonderwijs, aangezien hij geen principiële verschillen zag tussen inzicht in meetkunde en inzicht in wiskunde in het algemeen." (p190)

[3] My translation of: "De hier voor de meetkunde aangegeven weg kan nl. ook voor andere kennisgebieden gebruikt worden." (p202)

In their memorial text Broekman & Verhoef (2012:123) refer to Van Hiele (1964), a contribution in German to Odenbach (ed) 1964: "In the background there was the struggle by Van Hiele with working with the two different intuitions that already could be seen with Pythagoras and other ancients. When children learn, this can be seen in the intuition for continuity and that for discreteness, as this shows up in (spatial) geometry or counting (with integers) respectively. That discreteness concerns the transition from experience to abstractions in the form of symbols - thus to detach yourself from the image that is experienced and that determined the 'number'. In view comes arithmetic, numbers are nodes in a large network of relationships [reference]." [4]

All this should not surprise us. The Van Hieles started with Piaget's theory, which was a general theory of development. Their alternative was another general theory of development. They only took geometry as their test case for their theses.

W.r.t. the problem of induction: it is hard to prove a theory for all disciplines, even those not invented yet. The Van Hieles were aware of the limits of empirical methodology. See however Colignatus (2011c) for the "definition & reality methodology": definitions will be chosen such that reality can be covered with minimum uncertainty, and in some cases we might achieve virtually zero uncertainty. It is interesting to observe that Van Hiele (1957:191) was actually rather aware of this too: "We have mentioned already how the result of a study is often largely established by the choice of the definition." [5]

Diagram, embodiment and abstraction

I do not intend to review Tall's book here, only to set the record straight w.r.t. Van Hiele. But perhaps a general remark is allowed. At points the reader can embrace Tall's objectives, yet at other points one wonders whether he has actually used Van Hiele's work in practice. Too often we see Tall perform as a mathematician trained for abstraction and too often we don't see Tall perform as an empirical scientist who has recovered from his training for abstraction. My diagnosis is that Tall (2013) is a seriously misdirected book and needs a full rewrite. I will explain this in more detail at another place. Let me now consider the issue of Tall's diagram of the "three worlds of mathematics", and the issues of embodiment and abstraction, as they relate to Van Hiele's levels of understanding of mathematics (i.e. abstraction).

Tall (2013:17-19) - see his online PDF of the first chapter - presents a diagram (or table) in which we can recognise the Van Hiele levels. Tall's new format introduces separate

[4] My translation of: "Op de achtergrond speelde hierbij de worsteling van Van Hiele met het werken met twee verschillende intuïties die ook al bij Pythagoras en zijn tijdgenoten te onderkennen waren. Dit komt voor lerende kinderen tot uiting in de intuïtie van continuïteit en die van discreetheid zoals die naar voren komen in de (ruimte-) meetkunde respectievelijk het (met gehele getallen) rekenen. Die discreetheid heeft betrekking op de overgang van aanschouwelijkheid naar abstracties in de vorm van symbolen — dus het zich losmaken van het aanschouwelijke beeld dat het 'aantal' bepaald heeft. Het rekenen komt in zicht, getallen zijn knooppunten geworden in een groot relatienet [reference]."

[5] My translation of: "Wij hebben er al eerder op gewezen, hoe door de keuze van de definitie het resultaat van het onderzoek dikwijls al grotendeels vastligt."

attention for the senses, notably vision and sound (language, symbolics). In principle also touch and motion would be important but this might be taken along in "language". One might test the aspects by using the vision and sound buttons on the tv-control, on broadcasts with or without subtitles. We essentially see the two hemispheres of the brain, with the prefrontal cortex monitoring. It might lead too far but it has been suggested that Greek culture was visual and Oriental culture was aural, so that we could understand Euclid's *Elements* as a result from the clash of civilizations in Alexandria: to write down in linguistic legal fashion what the visual mind could perceive. Overall, it seems a useful idea of Tall indeed to use the visual method of a diagram in two dimensions to display the field of discussion. Overall, this should be used with caution however, since symbolics like Roman XII or Hindu-Arabic 12 might be attributed to "language" but clearly have visual aspects. Problematic in Tall's diagram is his treatment of abstraction and his use of the term "embodied".

Above, we quoted Van Hiele (2002:28): “People applied terms such as ‘abstraction’ and ‘reflection’ to the stages leading from one level to the next. This resulted in a confusion of tongues: we were talking about completely different things.” Van Hiele means that levels in understanding, with their web of relations of concepts, and actually diffferent meanings of the same words (actually speaking another language), cannot be merely reduced to such a vague term as "abstraction". When this is clear, I however would like to suggest that it can be advantageous to refer to the *levels of understanding of mathematics* (insight) as *levels in abstraction*. These are all somewhat vague notions while there is good reason to regard "abstraction" and "mathematics" as equivalent if not identical.

It is important to emphasize that thought is abstract by nature. When an apple creates an image in the mind (with all available senses, not just visual), then this mental image is abstract, and the brain can start processing it. For the education of mathematics it is crucial to be aware of this abstract nature of thought. It was the error of Hans Freudenthal to misunderstand the basic Van Hiele level: mistaking Van Hiele's reference to concreteness for some kind of "experience of reality", and even derive the name of his "realistich mathematics education" from it, and to introduce all kinds of real-world aspects into the curriculum and lose sight of that essential abstract nature of mathematics. It is the challenge for mathematics education to get pupils to focus on the abstract aspects that teachers know are useful to focus on. Colignatus (2011a) section 15.2 discusses how an essential step can be made here, in the "Conquest of the Plane". Colignatus (2011b) has some comments w.r.t. brain research with this role for abstraction.

Let us first present the Van Hiele levels of abstraction (insight) and then look at Tall's diagram. It is a sobering thought that we are basically classifying the math subjects of elementary school through university, but the thrust of the Van Hiele levels is the associated didactics, of providing the pupils with the appropriate materials and instructions. I feel a bit ashamed of presenting the table but the confusion by professor Tall requires an answer, and the point remains that one better considers the wealth in the research by the Van Hieles for educational practice.

Table: Van Hiele levels of abstraction (insight) distinguished by brain function

Deductive	*3. Formal*	Euclid		Hilbert
	2. Informal		School analytic geometry	School algebra
Practical	*1. Description*	Elementary school Geometry	Overlap, such as procept	Arithmetic
	0. Intuition	Visual emphasis	Overlap v & a	Aural emphasis

In the basic Van Hiele level, the pupil is mainly sensing the world. Basically the outside world feeds the memory via the senses, as the memory itself isn't well developed to feed the mind. The pupil learns to recognise objects and the space in which the objects occur. My preference for the basic Van Hiele level is the term "intuition" as it indicates a key role for the subconscious mind as opposed to conscious cognition at the higher levels (even though the interaction between these two is more complex than passing on of sensory data). The point isn't quite this experience of reality, as Freudenthal might suggest water running from a faucet as the experience of a linear process. The point is that the experience is concrete, like a line drawn with a ruler, so that the process of abstraction has traction to start from something (close to the intended mental image). Van Hiele has the opposition concrete vs abstract, Freudenthal model vs reality, see Colignatus (2014c).

For level 3, the lawyers at the department of mathematics might argue that Euclid isn't quite formal enough to pass for the claim of being "formal" conform David Hilbert, since his definitions and axioms refer to notions in the "intended interpretation" of visual space. However, instead of including additional levels (as originally by Van Hiele), it remains a good idea of Tall to include these extra columns. Then, though Euclid contains a lot of words, he is conveniently put in the column with the visual emphasis.

Perhaps superfluously, it may be remarked that aspects of logic and set theory and notions of proof should already be taught at elementary school. It is the failure of mathematics educators with the Sputnik "New Math" and "realistic mathematics education" (RME), and so on, that causes that we do not develop what is potentially possible. In practice education at elementary school already uses notions of proof as it isn't all rote learning, so that above distinction between "deductive" and "practical" is again one of emphasis and degree and level of abstraction.

Let us now look at Tall's diagram. Tall (2013:17)'s "three worlds of mathematics" are: "One is based on (conceptual) embodiment, one on (operational) symbolism, and the third on (axiomatic) formalism, as each one grows from earlier experience." He also uses "structural, operational and formal abstraction". (Check the Tall (2013:153) quote above.)

Diagram: "Figure 1.5" copied from Tall (2013:17)

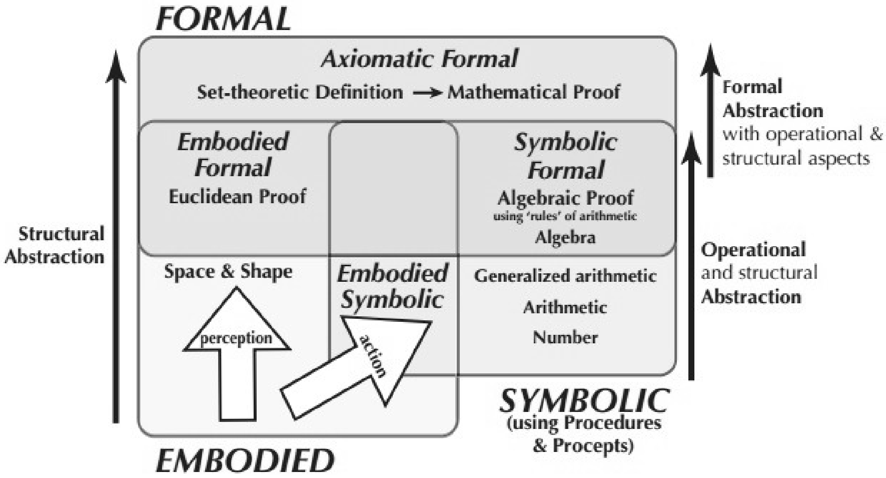

Figure 1.5: Preliminary outline of the development of the three worlds of mathematics

Professor Tall proposes to use the term "embodied". This term however has an inverse meaning w.r.t. the Van Hiele level. Van Hiele considers the situation that objects of the outside world create images in the mind of the pupil. An inverse process starts with an idea. We say that *an idea is embodied in something*. Tall must reason as a mathematician for whom a soccer ball that the pupil plays with is an embodiment of the mathematical idea of a sphere. Perhaps Tall's use of the word "embodied" is acceptable for objects, but it becomes awkward when the pupil learns the properties of empty space, (e.g. that a meter is the same in any direction); but we can stretch the meaning of embodiment too. In that case physical space embodies some math space. (Even though math space is *empty* by abstraction, and "embodiment" of *nothing* is a difficult notion to grasp, especially as some suggest that physical space isn't empty.)

Curiously though, Tall gives a definition of embodiment that focuses on the mental image, as in "the word became flesh", e.g. the pupil's mental concept of the soccer ball. Tall (2013:138-139) discusses President Bush senior's "decade of the brain" with various studies, and concludes: "The framework proposed in this book builds from sensori-motor operations into *conceptual embodiment* focusing on the properties of objects and *operational embodiment* focusing on operations, using language to describe and define more subtle forms of reasoning." (my italics)

The basic Van Hiele level uses the world as external, where the world supports human memory by providing the input to the senses, so that the mind can start collecting, memorizing, categorizing and so on. For didactics it is important to be aware what can be done at this level. In fact, in various other places Tall (2013) uses "embodied" in that external sense in various places, for example in tracing a curve by the fingertops to understand "tangent". If embodiment is external, then terms as "conceptual embodiment" and "embodied formal" are contradictions in terms.

If Tall intends embodiment to be internal (representation in neurons and chemicals) then he would lose contact with what Van Hiele proposes for the base level, and the didactics there. This kind of embodiment also becomes rather vague, without an operationalisation that a neopositivist would require - but which brain researchers might look for, such as Dehaene on numbers - and without much help for didactics that doesn't use brainscanners.

Tall is struggling with his terms overall. On p133 he refers to "thinkable concepts" but please explain "unthinkable concepts". On p425 he attributes his notion of "crystallization" / "crystalline" to a discussion with Anna Sfard, and on p 429 to a discussion with Koichu and Whiteley. It appears to be a rephrasing of Piaget's "encapsulation" and Van Hiele's compacting of various properties to a more unified concept. One better rejects that term "crystalline" since, taking a neopositivist stand again (though not in principle), it suggests more than there is. Are George's "crystalline" concepts the same as Harry's ? The phrase adds nothing, while of course the phrase suggests that Tall adds something to the discussion which on this point is not the case, as he only rediscovers what Piaget enlightened and what the Van Hieles systemized with proper empirics. (Van Hiele regretted the lack of empirical testing on his theory, but it remains empirical, and the evidence is supportive.)

Finally, Tall suggests the notions of *structural, operational and formal* abstraction, thus again relating to the two brain hemispheres and the prefrontal cortex, and relating to the new columns introduced with the new diagram / table for the Van Hiele level structure. Tall is careful enough to say that these types of abstraction may be difficult to distinguish. My problem with this is: (a) this may distract attention from the fact that the Van Hiele levels already concern abstraction, (b) this might come with the suggestion of something new but there isn't anything new, except for the words, (c) the distinction in types of abstraction creates an illusion of exactness.

Tall merely introduces new words to describe what happens in the Van Hiele levels. Van Hiele recognised that there can be a level shift in the step from arithmetic to proper algebra. He already knew that this was different from a level shift in geometry. What is the use to label the first as "operational" and the latter as "structural" abstraction ? And, confusingly, doesn't algebra contain some "structure" too ? Or doesn't geometry contain "operations" too (like drawing a circle with a compass) ? The "issue of distinguishing kinds of abstraction" is only created by the inclusion of the visual / aural columns, but this should not distract us. Overall, the use of new words may also be a matter of taste. The key point remains that there should not be the suggestion of something new on content.

Van Hiele rejected the vague word "abstraction". But once the Van Hiele levels in insight in mathematics are properly understood, I agree with Tall, and already proposed independently, that it can be advantageous to redefine the issue in terms of "abstraction" anyway, namely given that thought is essentially abstract and given that mathematics is equivalent if not identical to abstraction.

Tall's view on Pierre van Hiele and Hans Freudenthal

Setting the record straight w.r.t. Van Hiele as presented by Tall also causes a look at the influence of Freudenthal. If Tall did not fully appreciate the work by Van Hiele, might there have been an influence from what Tall read from Freudenthal's description of that same work ? Thus, while we focus in this article on VH & T, there is also F, generating the relationships VH & F and T & F. If the relationship VH & F was fair and F reported correctly, then this report would not have been a cause for T's long misunderstanding of VH. But if something went wrong in T & VH and the reporting was biased, then Tall might be wrongfooted, as the rest of the world.

In Freudenthal and Van Hiele we have two Dutch researchers with some impact on mathematics education on the world stage. It has some interest, both on content and history, what an outsider like professor Tall observes on this, and how this affected and affects his appreciation of their works. Note that the commission on math instruction of the international mathematics union (IMU-ICMI) has a "Hans Freudenthal Award" rather than a "Piaget - Van Hiele Award". We may somewhat infer that Freudenthal's international influence seems greater. An international reappreciation of Van Hiele might cause an reconsideration though.

Let us thus focus now on Tall's perception and presentation of the Van Hiele and Freudenthal combination. As said, Tall is a foreigner and outsider to this, while the insider Dutch have the advantage of additional personal information, documents and e.g. newspaper articles in Dutch, but perhaps the disadvantage of missing the bigger picture. My own position comes with the advantage of distance in time, as I came to teaching math at highschool only in 2007. I have the (dis-) advantages of being Dutch and a foreign exchange highschool student year in California 1972-1973. Also my first education was a degree in econometrics in 1982 and my degree in teaching mathematics came later in 2008. My background in empirics differs from a first training in abstraction only, as happens with mathematicians.

As William Thurston (1990, 2005) and Hung-Hsi Wu (see Leong (2012)) complain for three decades about the dismal state of math education in the USA, one should hope that there are independent factors at work in the USA itself, but the influence from Freudenthal with his advocacy of "realistic mathematics education" (RME) should not be regarded as negligible. When Thurston submitted his (1990) text to arXiv, he added this comment (2005): "This essay, originally published in the Sept 1990 Notices of the AMS, discusses problems of our mathematical education system that often stem from widespread misconceptions by well-meaning people of the process of learning mathematics. The essay also discusses ideas for fixing some of the problems. Most of what I wrote in 1990 remains equally applicable today."

A key document to understand the Van Hiele - Freudenthal combination is the interview in Dutch with Van Hiele by Alberts & Kaenders (2005), in the mathematics journal NAW of the Dutch Royal Mathematical Society (KWG). There Pierre van Hiele says: *"Trouwens, Freudenthal heeft mij later nogal eens een hak gezet, jongens."* (p247). This is Dutch idiom. Google Translate July 26 2014 gives literally: "Besides, Freudenthal has often put a heel to me later, guys." My proposed free but clearer translation, taking account the rest of the interview, is: "Besides, Freudenthal has later frequently sabotaged my work, guys." It is important to add that Van Hiele remained polite, as a fine math teacher would do.

Broekman & Verhoef (2012:123), in their short biography after the decease of Van Hiele in 2010, confirm that Van Hiele would have appreciated a university research position but wasn't offered one, and thus remained a highschool teacher all his life. The reason is not that such positions could not have been made available, or even a professorship. Broekman & Verhoef describe differences of opinion between Freudenthal ("reality vs model") and Van Hiele ("concrete vs abstract") but they do not mention the crucial distinction that Van Hiele had an empirical attitude while Freudenthal remained locked in mathematical abstraction (with a virtual notion of "reality").

On my weblog, I have concluded that Freudenthal's "realistic mathematics education" (RME) (i) partly doesn't work and (ii) that the part that works was mostly taken from Van Hiele. Freudenthal's "guided reinvention" is Van Hiele's method of providing the students with the relevant materials and instructions so that they can advance in the levels of understanding. In World War 2 in 1940-1945 Freudenthal was in hiding, taught his children arithmetic, relied on real world examples, and wrote a notebook on this. Van Hiele provided theoretical justification for the concrete vs abstract distinction. The name RME essentially refers to the basic Van Hiele level. Though Freudenthal will have been inspired by his own experience, he must have realised that the scientific basis was provided by Van Hiele, and thus his choice of the name "realistic mathematics education" amounted to some appropriation and distortion of the Van Hiele result. While Freudenthal at first referred to Van Hiele he later tended to refer to "his own publications", which had the effect that Van Hiele wasn't openly referred to. Colignatus (2014a) concludes that RME is a fraud. These conclusions on Freudenthal himself and RME directly affect our understanding of how Tall and the rest of the world were disinformed, which helps our understanding of Tall's perception of Van Hiele's work. My weblog short discussion and conclusion clearly only touch the surface of the problem. One can only hope that funds will become available both to analyse the errors of the decades since 1957, and to develop ways to repair those.

To put the issue in context: Colignatus (2014b) concludes that there is a serious issue with scientific integrity in the Dutch research in mathematics education, starting with Freudenthal but now with a dysfunctional "Freudenthal Institute" with a loyality complex, and with also failing supervision by the Dutch Royal Academy of Sciences (KNAW). The same disclaimer w.r.t. a weblog article applies. Those conclusions on the currently dysfunctional "Freudenthal Institute" are of less direct relevance for our topic of discussion of getting the facts right on Van Hiele & Tall. However, it is useful to mention them since they put the issue in context. For example, professor Tall indicated (in an email conversation) that it was hard to find English sources of Van Hiele's work. Part of the explanation is not only in the observable sabotage by Freudenthal himself but also in the continued similar dysfunction by said institute, that should have been able to recognise the importance of Van Hiele's work and help make translations available.

After I wrote that analysis on my weblog (2014a), a Dutch reader drew my attention to the thesis in Dutch by La Bastide-Van Gemert (2006) (further LB-VG) that I had not seen yet at that time. This thesis is explictly on Hans Freudenthal on the didactics of mathematics, and her chapter 7 discusses Freudenthal and the Van Hiele levels. At first I was inclined to neglect this hint and study since it seemed that it did not pertain to this present paper on Van Hiele & Tall. But checking it, it appeared quite relevant. I see my weblog analysis confirmed. Above, below and in the **Appendix B** I provide some English translations of some passages. LB-VG describes on p195 that Freudenthal's work on math education was rather bland and traditional before 1957 and only gains content after the work by the Van Hieles. She decribes how Freudenthal first refers to the Van Hieles but then introduced new terms like "guided re-invention" and "anti-didactic inversion", and subsequently moved Van Hiele into the background and started advocating those own terms. However, LB-VG apparently falls for the suggestion that Freudenthal really contributed something new, and not just new phrases and misunderstandings. Given this (wrong) perspective, she does not discuss the scientific integrity problem of misrepresentation, appropriation of work and the withholding of proper reference. With the proper perspective her thesis however provides corroboration.

Above, I already quoted some evidence from her thesis concerning Pierre van Hiele's early awareness of the portent of his theory. There are some more particulars on Van Hiele and Freudenthal that might distract here and thus are put into **Appendix B**. A conclusion there is: While Freudenthal took key parts from Van Hiele's theory, he also inserted his own phraseology, with such consequence that Tall apparently had difficulty recognising Freudenthal's texts as Van Hiele's theory in (distorted) disguise, so that Tall could embark on his own path to re-invent Van Hiele's theory.

How does professor Tall deal with this situation of which he has been a foreign observer for all these decades ? Tall (2013:414-415) has a short text about 'realistic mathematics education' (RME). Tall (2013) refers to Van den Heuvel-Panhuizen (1998) of the "Freudenthal Institute", henceforth VdH-P. It is useful here to also refer to an early review by Tall (1977) of Freudenthal's book *"Mathematics as an educational task".* (Note that I regard mathematics as abstract and education as empirical, so that Freudenthal's book title reads to me as a contradiction in terms.) Thus, Tall (2013:414-415) on RME:

> "The Dutch project for 'realistic mathematics education' was introduced to build on the learner's experience and to replace an earlier mechanistic system of teaching routine procedures. [footnote reference to VdH-P] It provides the child with a realistic context in which to make sense of ideas that are often performed in a practical situation. Yet, as time passed, it was found that, at university level in the Netherlands, remedial classes needed to be introduced because more students lacked the necessary skills for advanced work in mathematics and its applications." [footnote reference to "Information supplied by my colleague Nellie Verhoef, based on articles in Dutch: (....)]
>
> The Dutch translation of the verb 'to imagine' is 'zich ***realise***ren', emphasizing that what matters is not the real-world context but the realization of *the reality in the student's mind,* which Wilensky expressed as the personal quality of the mental relationship with the object under consideration. [footnote reference]
>
> The three-world framework not only sees practical mathematics related to real-world problems, but it also offers a theoretical framework to realize ideas in a conceptual embodiment that transcend specific examples and blend with flexible operational symbolism."

We can deconstruct this quote on the points of (i) translation, (ii) the rewriting of history, and (iii) textual gibberish. Incidentally, Van Hiele (1957) distinguishes between students who rely on algorithms and students who are able to recognize structure. Apparently the reference to "mechanistic" by VdH-P concerns an emphasis in teaching upon algorithms rather than insight. But the words algorithm, structure and insight are avoided.

Firstly, Tall is erroneous on the Dutch translation:

- The translation of English "to imagine" to Dutch is "zich verbeelden, zich inbeelden, zich voorstellen" (as alternatives with different shades of meanings) and *not* "zich realiseren". I checked that Google Translate July 26 2014 had it good.
- The translation of Dutch "zich realiseren" (note the "zich"= "oneself") is "to become aware", as "He realised that the train would depart without him" or "She realised that a triangle with two equal sides also has two equal angles". Or see above quote of Tall (2013:430) how he realized only at the death of Van Hiele how important his contribution had been (except that Tall regards this as his own discovery and not something that Van Hiele had already been aware of and explaining about).
- It is not *making-real* as in "He realised his plan to reach the top of the mountain". In Dutch "Hij realiseerde zijn plan" has such meaning. The difference comes from "zich" (oneself) between "realiseren" en "zich realiseren".

Tall's erroneous translation of "to imagine" to Dutch apparently is based upon the erroneous translation provided by VdH-P (1998). This VdH-P text is problematic in various respects. It is useful to realise that there already was quite some criticism on RME in 1998, so that VdH-P's text also has a quality of defense against that criticism. It also rewrites history and presents a curious view on education. Apparently unwittingly, Tall is dragged along in this. VdH-P (1998):

> "The present form of RME is mostly determined by Freudenthal's (1977) view about mathematics. According to him, mathematics must be connected to reality, (...) It must be admitted, the name "Realistic Mathematics Education" is somewhat confusing in this respect. The reason, however, why the Dutch reform of mathematics education was called "realistic" is not just the connection with the real-world, but is related to the emphasis that RME puts on offering the students problem situations which they can imagine. The Dutch translation of the verb "to imagine" is "zich REALISEren." It is this emphasis on making something real in your mind, that gave RME its name. For the problems to be presented to the students this means that the context can be a real-world context but this is not always necessary. The fantasy world of fairy tales and even the formal world of mathematics can be very suitable contexts for a problem, as long as they are real in the student's mind."

Deconstructing this:

(1) First of all, the true origin of the "realistic" in RME derives from Freudenthal's real-world linkage and not from the "zich realiseren" (become aware). LB-VG (2006:194) confirms that Freudenthal in persecuted hiding during the war years 1940-1945 already wrote a didactic text for teaching arithmetic to his children, in which real life issues are prominent. This also relates to the switch from abstract Sputnik "New Math" to applied mathematics around 1970. But the theoretical justification for the concrete vs abstract opposition and the start from mere intuition only derives from the

Van Hiele theory. Van Hiele better explains than Freudenthal that teaching should start from what students understand, and that teaching is guiding them towards what they don't understand yet. The overall conclusion is that Freudenthal took the RME name from the Van Hiele basic level, as that had received theoretical justification.

(2) Secondly, the "mostly" and "not just" are a rewriting of this history into another interpretation, in which the "zich realiseren" is plugged in, taking advantage of the flexibility of language. There seem to be earlier occurrences of "zich realiseren" (possibly Gravemeijer 1994) before this present use by VdH-P, but this should not distract from the effort at rewriting history w.r.t. point (1) after 1957.

(3) Thirdly, while the Dutch term "zich realiseren" has an etymological root to "reality", the subsequent explanation by VdH-P of "zich realiseren" should be about "to grow aware" (which is the proper translation). However, we see that the explanation that she gives is about the ability to understand (imagine) the situation under discussion. While history is rewritten, this new interpretation of "realistic" has a different meaning in Dutch (to grow aware) than in English (to be able to imagine). A wrong translation is deliberately used to suggest that the meaning would be the same in both languages.

(4) On content, I fail to understand why it would make a difference whether a student sees a fantasy **as** *the fantasy that it is* **or** *experiences it 'as real'*, if you want to link to their existing stock of experience and mental abstraction to start doing mathematics. The suggested didactic condition "as long as they are real in the student's mind" is unwarranted. (For example, if one presents a fantasy cartoon image of a blue car and a photograph of a red car, and starts a discussion whether one can do 1 car + 1 car = 2 cars, we may suppose that part of the discussion would be about differences between cartoons and photographs, and that you cannot add images to real cars on the parking lot. But you can count cars on the parking lot and add their numbers and use images to represent them, even fantasy cartoons and "realistic" photographs.)

Subsequently, Tall's phrase "realize ideas in a conceptual embodiment" is gibberish. The "realization of ideas" is ambiguous, as said. Is it growing aware or is it making-real ? Students can draw lines and circles, to approximate abstract ideas, but we cannot assume that they can turn cartoons of cars into actual cars on the parking lot. To grow aware of ideas is excellent, but this requires the Van Hiele teaching method and not the distorted versions. The "conceptual embodiment" is, as said, either a contradiction in terms, or it refers to a constellation of neurons and biochemicals, but then yet lacks an operational meaning useful for teaching.

Taking Tall's quote as a whole, he seems to suggest that his confused approach would be helpful for the Dutch situation that pre-university teaching has gone haywire and requires remedial teaching at university. His frame of reference prevents him from observing that the error lies in RME and in the work of the very VdH-P whom he quotes. It prevents him from "realizing" that he himself is also off-track in his book and in joining up with a failing Dutch constellation.

What really happens here is that professor Tall apparently wants to link up to the Dutch situation and RME theories, perhaps wanting that his theory is also accepted at the dysfunctional "Freudenthal Institute", without having adequate understanding about the local maltreatment of Van Hiele's work. The information that he relied on by VdH-P was wrong. The information that he quoted as receiving from Verhoef seems incomplete, and last week I got evidence that criticism is possible on subsequent issues. Who in Holland will defend Pierre van Hiele, with the indicated loyality complex at the "Freudenthal Institute" ? I informed Tall about my books Colignatus (2009) and (2011a). Professor Tall and I actually met at a conference in Holland on June 24 2010 and spoke a bit longer. In an email of 21 May 2012 he mentioned that he was in the autumn of his life and wished to get his book finished, and didn't have time to look at my books. It is a regrettable paradox that the amount of time taken to write is very much more than the amount of time needed to see that one should write something else and much shorter. Still, one can only respect a person in the autumn of his life. In a way I consider it very useful that Tall has taken stock of his work, since it shows both a misunderstanding of Van Hiele over most of Tall's life and a recognition of the importance at that autumn. Hopefully Tall continues to think about math education, hopefully also in an essential rewrite.

Conclusion

It is somewhat enlightening to conclude that professor Tall as researcher and teacher in the education of mathematics seems to have little experience in the use of Van Hiele methods in actual educational practice. In itself it is relevant that Tall recognises the importance of Van Hiele's work at this late stage, in retirement. It remains also a phenomenon to be explained that such an important theory recieves such recognition by Tall only at such a late stage. For this, the details of the Van Hiele - Freudenthal combination are relevant, of which researchers in education in mathematics do not appear to be aware about in general.

David Tall grew aware to a much larger extent in 2010 of the Van Hiele niveaux of understanding of mathematics. He also thought that Van Hiele (1957, 1959, 1973, 1986, 2002) saw only limited application. Tall now claims that it was a creative insight on his part to extend those levels to wider applicability. Perhaps it was, given his misunderstanding of Van Hiele. Perhaps it was only a recollection of something read or heard but forgotten and surfacing in different form. Whatever this be, the claim however does no justice to Pierre van Hiele who already asserted that wider applicability, also for other disciplines than mathematics, in 1957. Tall's claim may block researchers in education in general and the education of mathematics in particular from considering the wealth in Van Hiele's work. We owe Pierre and Dieke van Hiele and our students to get the facts right.

Thomas Colignatus is the science name of Thomas Cool, econometrician (Groningen 1982) and teacher of mathematics (Leiden 2008), Scheveningen, Holland.

I thank professor David Tall for graciously even though critically informing me and providing details so that I could better understand his analysis and position. I have offered to write a joint paper on this issue so that others would not be in doubt concerning his reaction to this information that apparently is new to him. Given his initial rejection of this suggestion I deem it better to clearly state that information.

I also thank professor Jan Bergstra (Amsterdam, KNAW) for providing critical comments and for drawing my attention to La Bastide - Van Gemert (2006). As the latter thesis focuses on Freudenthal I at first considered it not relevant for the present focus on Van Hiele and Tall, but it is another indication of the interrelatedness of things that it appeared to contain very relevant evidence.

Appendix A: Basic data and problems of translation

(1) Pierre van Hiele's thesis advisor was H. Freudenthal while Dieke van Hiele's thesis advisor was M.J. Langeveld. The main source is http://dap.library.uu.nl. These data are also in http://genealogy.math.ndsu.nodak.edu/id.php?id=102372 and id=144944. Some other sources give conflicting information:

1. The "Freudenthal Institute" has Langeveld as supervisor for Pierre but this is only correct for his role as second supervisor: http://www.fisme.science.uu.nl/wiki/index.php/Pierre_van_Hiele
2. Broekman & Verhoef (2012:123) state: "Van Hiele himself mentioned at more occasions, in private and not in public, that he interpreted that Freudenthal's choice to be the first supervisor of Dieke, which implied that he himself 'got' Langeveld as first supervisor, also indicated that Freudenthal rejected his more theoretical work (on the psychology of cognition and learning)." [6] This remark by Pierre thus would refer to the early period before the final thesis advisors were allocated.

(2) Professor Tall alerted me to Fuys et al. (1984) *"English Translation of Selected Writings of Dina van Hiele-Geldof and Pierre M. van Hiele"* downloadable at ERIC. Page 8 states that the translations have been accepted by Pierre van Hiele.

This source puts an emphasis on geometry, possibly stimulating the confusion for some authors that the Van Hieles might think that the levels applied to geometry only. Indeed, the translations were produced under the project title: *"An Investigation of the van Hiele Model of Thinking in Geometry among Adolescents".* Such a positioning might run the risk of excluding the wider scope. However, Dieke's chapter XIV *"Further analysis and foundation of the didactics"* looks at general didactics, in particular p202 when she compares with her own learning process in didactics. Importantly, the "tenets" (p220-2) start with a general scope (e.g. "I. In order to be able to arrive at an efficient study of a certain subject (...)".

Secondly, accepting a translation is another issue than finding a translation that reduces confusion. A key point is the translation of the title of Pierre van Hiele's thesis:

- My translation: "The Issue of Insight, Demonstrated with the Insight of School Children in the Subject Matter of Geometry." [7]
- The ERIC p8 translation, apparently accepted by Van Hiele: "The Problems of Insight in Connection with School Children's Insight into the Subject Matter of Geometry".

I have these considerations for my translation: (a) Keeping the original "demonstrated" in the title warrants that geometry does not only provide an existence proof for the levels but also forms only an example. (b) The reference to the role of demonstration in geometry itself must have been quite deliberate. The thesis discusses that mathematics is about

[6] My translation of: "Zelf heeft Van Hiele in besloten kring meerdere malen het idee geopperd dat de keuze van Freudenthal om eerste promotor te zijn van Dieke, waardoor hijzelf Langeveld als eerste promotor ‘kreeg’, door hem altijd beschouwd is als een afwijzing door Freudenthal van zijn meer theoretisch (denk/leerpsychologisch) werk."

[7] My translation of: "De Problematiek van het Inzicht, Gedemonstreerd aan het Inzicht van Schoolkinderen in Meetkunde-leerstof". (The ERIC text on p258 suffers from typing errors.)

proof, after all, and not just the execution of algorithms. (c) Van Hiele (1957) and (1959) emphasizes the triad teacher-student-subject. Thus, while the theory of levels is general, the subject matter exerts a relevant influence for particular educational situations.

PM. Some smaller comments on Van Hiele (1959): (a) Van Hiele speaks about five levels and at first only presents four, but eventually the fifth appears on page 254 (using the ERIC page count). (b) It remains awkward that Van Hiele gave the label 0 to the lowest level. The ranking words "first" to "fifth" tend to become ambiguous, as they apparently also might function as adjectives (associating "the third level" with "level 3"). (c) The article is a translation from French, in which the word "intuition" might have been used for "insight" ? Overall, it might be better to call the base level the "intuitive level", where insights are still unguided (while there may of course be trained intuition at higher levels).

Appendix B: Additional information

La Bastide-Van Gemert (2006) (further LB-VG) gives additional evidence on Freudenthal and the Van Hiele level theory. Her text apparently requires some deconstruction however since she appears to have a rather rosy view on Freudenthal's performance.

The body of the text above contains her quote of the *Euclides* 1957 article with the statement by the Van Hieles that the theory of levels applies to other disciplines too.

LB-VG then arrives at this curious statement:

p197-198: "In that manner Freudenthal described the theory of levels by which the direct link with geometry, essential and explicitly relevant in the work by the Van Hieles, had disappeared. It seemed that Freudenthal by this abstraction started to see the theory of levels as independent of the context (geometry education) under which he had learned about it: a new level had been reached ...." [8]

Comment: (a) The suggestion of a direct essential link with geometry is inconsistent with the earlier observation that geometry was only an example. (b) LB-VG suggests that the thesis supervisor Freudenthal would not have seen that. (c) There is the false suggestion that only Freudenthal made that step into abstraction toward general application.

Just as curious, p198: "In that manner, step by step, Freudenthal gave his own interpretation of the theory of levels. Independent from the education of geometry from which the theory originated, he abstracted it into a method of logical analysis in the clarification of the (levels in the) educational topic of interest." [9]

Comment: Apparently she is not aware of the inconsistency, and would not quite understand what Van Hiele had achieved.

p199. Another point, of which Van Hiele will have been aware, but which apparently was also claimed by Freudenthal, and again by Tall (2013), chapter 14, but apparently new to LB-VG (2006): "In modern mathematics the mathematical systems, that have arisen by the organisation and ordering of the topic of interest (the relevant mathematical issues), became the subject of organisation themselves, via axiomatisation. We find a remarkable parallel between this remark [by Freudenthal] and Freudenthal's interpretation of the theory of levels: in mathematics itself there were, in this manner, transitions (definable by

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[8] My translation of: "Zo beschreef Freudenthal de niveautheorie op een manier waarbij de directe link met de meetkunde, essentieel en nadrukkelijk aanwezig in het werk van de Van Hieles, verdwenen was. Het leek erop dat Freudenthal door deze abstrahering de niveautheorie los begon te zien van de context (het meetkundeonderwijs) waarin hij ze leerde kennen: een nieuw niveau was bereikt...."

[9] My translation of: "Zo gaf Freudenthal de niveautheorie van de Van Hieles stap voor stap een eigen invulling. Los van het meetkundeonderwijs waar de theorie uit voort kwam, abstraheerde hij deze tot een werkwijze van logische analyse bij het inzichtelijk maken van de (niveaus van de) leerstof."

logic) to a higher level, comparable to the transitions between the levels such as there appeared to exist within the process of education." [10]

While the above indicates that LB-VG would not have been at home in the Van Hiele theory of levels, the following in her chapter 7 gives evidence on other forms of scientific malconduct by Freudenthal:

- p191: A 1957 newspaper article quotes Freudenthal giving a wrong description of the levels (namely: in mastery of routine) - but such may happen with newspapers.
- p204: "The theory of levels as such disappeared in Freudenthal's publications into the background and he based himself primarily upon his own ideas such as 'anti-didactic inversion' and 'guided re-invention' that for him were related (implicitly or not) to this theory of levels. He still mentioned the Van Hieles and their work in his articles but now only in passing. For Freudenthal the work by the Van Hieles had been promoted to basic knowledge." [11] Comment: In my analysis, LB-VG takes a rosy view on this. Freudenthal must have known that the Van Hiele levels were not well-known, and certainly not their claim on wider application than geometry only. Instead, Freudenthal reduced Van Hiele to geometry only and advanced his own phraseology as the proper approach in general.
- p182 gives a quote by Freudenthal in his autobiographic book p354, which is rather convoluted and lacks the clarity that one would expect from a mathematician: "The process of mathematisation that the Van Hieles were mostly involved with, was that of geometry, more exactly put: they were the first who interpreted the geometric learning process as a process of mathematisation (even though they did not use that term, and neither the term re-invention). In this manner Pierre discovered in the educational process, as Dieke described it, the levels of which I spoke earlier. I picked up that discovery - not unlikely the most important element in my own learning process of mathematics education." [12] Comment: Freudenthal thus suggests: (a)

[10] My translation of: "In de moderne wiskunde werden de wiskundige systemen die ontstaan zijn door het organiseren en ordenen van het onderwerp (de betreffende wiskundestof) zélfonderwerp van organisatie, van axiomatisatie. Tussen deze opmerking en Freudenthals interpretatie van de niveautheorie is een frappante parallel te trekken: in de wiskunde was er op die manier sprake van (door logica definieerbare) sprongen naar een hoger niveau, vergelijkbaar met de sprongen tussen de niveaus zoals die er binnen het onderwijsproces bleken te zijn."

[11] My translation of: "De niveautheorie als zodanig verdween in Freudenthals publicaties naar de achtergrond en hij beriep zich voornamelijk op de voor hem (al dan niet impliciet) met deze theorie samenhangende ideeën als 'anti-didactische inversie' en 'geleide heruitvinding'. Hij noemde de Van Hieles en hun werk nog steeds in zijn artikelen, maar nu slechts en passant. Het werk van de Van Hieles was voor Freudenthal gepromoveerd tot basiskennis."

[12] My translation of: "Het mathematiseringsproces waar de Van Hieles zich vooral mee bezighielden, was dat van de meetkunde, preciezer gezegd: ze waren de eersten die het meetkundig leerproces als proces van mathematiseren interpreteerden (al gebruikten ze de term niet, evenmin als de term heruitvinding). Zodoende ontdekte Pierre in het onderwijs, zoals Dieke het beschreef, de niveaus waarvan ik eerder sprak. Ik pakte die ontdekking op – wellicht het belangrijkste element in mijn eigen wiskunde-onderwijskundig leerproces."

Pierre's insight is just seeing what Dieke described, so that she would be the real discoverer. (b) Freudenthal's words "mathematisation" and "re-invention" would be crucial to describe what happens in math education, otherwise you do not understand what math education is about, and it is only Freudenthal who provided this insight. (c) The Van Hieles wrote about geometry but were limited to this, so that it was Freudenthal himself who picked it up and provided the wider portent by means of his new words.

- p194, taking a quote from Freudenthal's autobiographic book p352: "What I learned from the Van Hieles I have reworked in my own manner - that is how things happen." [13] Comment: This is the veiled confession of appropriation. Freudenthal claims to be powerless and innocent of deliberate appropriation since "that is how things happen". Who however considers what that "reworking" involves sees only phraseology and lack of proper reference.
- p205: She discusses Freudenthal's use of "reflection" for the level transition, that Van Hiele (2002) protests about. As far as I understand this discussion, Freudenthal essentially merely provides introspectively, and without empirical support, the word "reflection" in relation to a level transition, as if only his new word is the valid approach, so that only he can be the inventor of transition via that proper word. However, the proper scientific approach would have been to describe what the Van Hiele theory and approach was, then define what the new idea of reflection would be, and provide the empiral evidence on that new insight (as the Van Hieles had provided empirical data for their method to achieve level transitions).

I might mention that LB-VG doesn't seem to be aware of Van Hiele's insight in the role of language. We could consider more points but it seems that the above suffices.

Another conclusion is: While Freudenthal took key parts from Van Hiele's theory, he also inserted his own phraseology, with such consequence that Tall apparently had difficulty recognising Freudenthal's texts as Van Hiele's theory in (distorted) disguise, so that Tall could embark on his own path to re-invent Van Hiele's theory.

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