Neoclassical mathematics for the schools

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Abstract

National parliaments around the world are advised to each have their own national parliamentary enquiry into the education in mathematics and into what is called 'mathematics'. Current mathematics education namely fails and causes extreme social costs. The failure can be traced to a deep rooted tradition and culture in mathematics itself. Mathematicians are trained for abstract theory but when they teach then they meet with real life pupils and students. Didactics requires a mindset that is sensitive to empirical observation which is not what mathematicians are basically trained for. The recent call by professor Wu to research mathematicians to start participating actively in the education enterprise (see the AMS Notices March 2011) calls for the wrong cavalry. We need engineers with an empirical set of mind rather than abstract academics. The mathematics required for schools likely can best be called "neoclassical mathematics" and is based upon the books "A Logic of Exceptions", "Elegance with Substance" and "Conquest of the Plane" by the same author of this article.

Introduction

If we want to improve the education in mathematics then we must consider the content, the education of teachers and the tools. Below gives an outline redefinition of the content into Neoclassical mathematics (NM). For the (re-) education of teachers we need the involvement not quite of research mathematicians but rather of the empirical sciences, since education is an empirical issue. The tools follow from these.

This point of view differs from the distinction by Hung-Hsi Wu [Wu1] & [Wu2] into Research mathematics (RM), School mathematics (SM) and Textbook SM (TSM). Wu estimates roughly that TSM may contain an error every two pages. Teachers get TSM as basic education and RM at higher education, and never really arrive at some ideal SM. [Wu1] is a call for action directed at research mathematicians to co-operate with the teaching community to actually create that SM and its (re-) education of teachers. Wu's call does not mean that only research mathematicians can help out, since also the education community has a stake. However, it is not a call to empirical science.

The distinction between these two views concerns empirics. The education community is insufficiently empirical and research mathematicians may help but might also do damage. Mathematicians are trained for abstract thought but pupils happen to occur in real life. Teachers try to resolve their cognitive dissonance by relying on tradition, but traditional mathematical content is a nightmare. The ideal SM that Wu paints still suffers from the very same blindness to reality. Creating more consistency into a nightmare does not remove the very nightmare itself. This will be illustrated below with a discussion on fractions. A longer exposition and many more cases can be found in "Elegance with Substance" [EWS] and "Conquest of the Plane" [COTP]. I admire professor Wu for his insights in and contribution to the education of mathematics. That even professor Wu falls into the trap of underestimating empirical science shows how difficult the subject is. We can only hope that the issue gets the best of our possible attention and therefor I advise each nation to have an enquiry by its national parliament. Mathematicians should be the first in line to ask parliaments to help them to carry the burden assigned to them of caring for the education in mathematics. Parliaments can be motivated by the properties of mathematics education: the costly investments in manpower and computer programs and equipment, as well as the level of education itself and the economic consequences.

The name "neoclassical mathematics" derives from the foundations of mathematics. We are familiar with the distinction between logicism, formalism and constructivism as those arose around 1900. Classical mathematics [CM] came into various problems. (1) The 'division by zero' of the derivative created historically the approaches of (a) exhaustion by Antiphon and Eudoxos, (b) infinitesimals by Archimede, Newton and Leibniz, (c) algebra by Euler and Lagrange, (d) limits by Cauchy and Weierstraß. (2) With the liar paradox of the ancient world there came the paradox by Russell and the theorems by Gödel. These issues (1) and (2) however are resolved by "A Logic of Exceptions" [ALOE], and see the review [RDG] in the Dutch journal of mathematics NAW, written by professor Gill of the Dutch Royal Academy of Sciences. Hence it is possible to teach mathematics again in quite classical perspective, using 2000 years of didactic advance as well of course. Research mathematics might continue with the neglect of [ALOE] but then would not put a burden on education (though possibly on financial markets and such). PM. An offspring of [ALOE] is "Conquest of the Plane" [COTP] with a favourable review by J.M. Gamboa of the European Mathematical Society [JMG]. The book "Elegance with Substance" [EWS] lies in time and purposes between [ALOE] and [COTP], and got a mixed review by G. Limpens [GL], who is critical of some aspects but in sum appreciates the critical look at mathematics itself.

PM. My suggestion is that the reform could be for K-12 but also the first year of college or university, but when the discussion takes place in the context of professor Wu's paper then it suffices to use the term “schools”.

We first consider the fractions and then give an outline of the neoclassical approach.

Fractions

(a) First consider 2½ for “two and a half”, where the position next to each other means addition. Secondly consider 2*a* for “two times *a*” or $2\sqrt{2}$ for “two times the square root of two”, where the position next to each other means multiplication. Comparing these, the positions next to each other thus are interpreted differently, and pupils must be trained to see the difference. This also causes that we must make sure that there is a space inbetween in 2 ½ when we want it to reduce to 1. This tradition of different interpretations of positions is curious, but it might be acceptable when we use typesetting with fixed places. The tradition however is asking for problems in handwriting when a pupil may write 2½ as 2 ½ or conversely, and thus slip into error. The solution is abolish the notation 2½ and to keep 2 + ½ so that the “+” nicely reflects the “and” in “two and a half” and so that the “+” may also be an end-station. This is similar to the case that $\sqrt{2}$ can be an end-station and need not be expanded in decimals 1.414… It takes a huge amount of time to train pupils now to write 2 + ½ as 2½ (and not reduce this to 1), and later again to unlearn this positional approach for 2*a,* and the only reason is tradition for tradition’s sake.

(b) [EWS] and [COTP] both present a *proportion space* and defend the point of view of Pierre van Hiele that kids at elementary school would be able to work with vectors and thus a *vector space*. These two spaces need an integral discussion otherwise there arises confusion.

(c) Another point is that division essentially links up with the algebraic approach to the derivative. Since Cauchy and Weierstraß we have been trained to focus on numerical aspects but Weierstraß already uses predicate logic and it appears that algebra and the logic of the manipulation of the domain create the derivative just as well. Even better, since this eliminates the paradox of ‘division by zero’ and it avoids the educational combi-load of both limits and the derivative. Limits and infinitesimals are useful, e.g. for the understanding of real numbers and approximations, but not necessarily for the derivative of functions used in K-12 (and likely wider). See [COTP] and also “Contra Cantor Pro Occam” [CCPO]. Hence, a good understanding of division is not only required to survive 3rd grade but also the derivative.

These insights (a), (b) and (c) are missing in [Wu3]. It merely illustrates the importance of the empirical approach to education, and may cause the reader to look at the other cases mentioned in [EWS] and developed in [COTP].

An outline

Neoclassical mathematics has no precise definition yet but uses [ALOE], with an application to education in [EWS]. The latter is implemented again in [COTP] with some comments in [CCPO]. Neoclassical mathematics gives a point of view that Aristotle and Euclid supposedly could live with, and that people might find rather natural to understand. Some points are:

(1) The liar and Gödeliar statements are nonsensical, in a three-valued logic.

(2) Russell's set paradox and Cantor's Theorem for infinite sets are nonsense too. We may use a set of all sets. There are no 'transfinites'.

For example, Russell's set is $R \equiv \{y \mid y \notin y\}$. This definition can be diagnosed as self-contradictory, whence it is decided that the concept is nonsensical. Using a three-valued logic, the definition is still allowed, i.e. not excluded by a Theory of Types (that makes it non-sensical too), but statements using it receive a truthvalue Indeterminate. An example of a set similar to Russell's set but without contradiction is the set $S = \{y \mid y \notin y \wedge y \in S\}$, which definition uses a small consistency condition, taken from Paul of Venice, see [ALOE] p127-129.

(3) Euclidean space is defined as our notion of space. Non-Euclidean space can only be imagined in Euclidean space.

(4) The natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$ are countable and a potential infinite, but their 'total' is an actual infinite. The continuum $\mathbb{R}$ or the interval [0, 1] is also an actual infinite. There is a bijection 'in the limit' between $\mathbb{N}$ and $\mathbb{R}$ such that these are 'equally large'. See [CCPO].

(5) Probability and statistics in relation to the sorites paradox.

(6) Mathematics (abstraction) and engineering (approximation to reality) are discussed in conjunction, to foster sensitivity to the translations. For example, measurement errors due to the constancy of the speed of light do not mean a distortion of space but remain measurement errors.

(7) An encyclopedia of mathematics, e.g. what might result if some assumptions are changed. For example two-valued logic, fuzzy logic, the Brouwer-Heyting axioms, incompleteness, computability, transfinites, fractals, chaos theory ...

(8) Democracy is a key concept but generally misrepresented by mathematicians, see my book "Voting Theory for Democracy" [VTFD]. Mathematician Kenneth Arrow claimed that reasonable and morally desirable properties caused an inconsistency and hence that what we ideally expect from democracy would be impossible. This however appears to be unwarranted. See [SRSR] how mathematicians are still locked in denial of truly reasonable analysis.

A table

	Traditional mathematics (TM)	Neoclassical mathematics (NM)
1	Two-valued logic. What is nonsensical is excluded by restrictions on form.	Three-valued logic. What is nonsensical is explicitly called nonsensical.
2	Gödels theorems on undecidability.	Under some stronger properties of the proof predicate the Gödeliar sentence causes a contradiction so that it can be judged to be as nonsensical as the liar sentence. There remains a similar kind of philosophy: that mathematical activity by mankind has the fundamental uncertainty that some inconsistency may pop up.
3	Zermelo Fraenkel axioms of set theory, also to deal with Russell's paradox	Selfreference is allowed, and nonsensical cases like Russell's paradox are recognised for what they are.
4	Cantor's Theorem on the subsets.	The theorem holds for finite sets but not for infinite sets. The diagonal argument appears to be nonsense.
5	Difference between countable and uncountable infinity. Transfinites.	Potential infinity associates with counting, actual infinity associates with the continuum. There is a bijection in the limit between natural and real numbers. No transfinites.
6	Weierstrasz for the derivative of regular functions (i.e. used in highschool).	Algebraic definition of derivative and integral for such functions. Limits are useful but not for the derivative. (Possibly Weierstrasz for other functions.)
7	What is 'space' depends upon axioms.	Euclidean space is defined as our notion of space. Non-Euclidean space can only be imagined in Euclidean space.
8	Arrow's Theorem shows that ideal democracy is impossible.	A key property of the ideal of democracy is that it should work. Hence one of Arrow's axioms has to be rejected. This appears to be the axiom of pairwise decision making.
9	Mathematics education for highschool and first year of college requires training on traditional concepts.	Mathematics education requires a fundamental re-engineering. Much of mathematical content will remain the same but there are key gains in consistency and didactics.

Conclusion

If neoclassical mathematics as indicated above is adopted as school mathematics then professor Wu probably still might be happy that there at least is a SM, and undoubtedly many kids would be happy too. The choices involved will be clear. When research mathematicians drop the nonsense and look more into engineering with sound standards, and when the empirical sciences look into the education in mathematics, then there will be more cause for hope for improvement. Since so much is at stake and since professionals entertain standards that currently cannot be met without sizeable investments, and since we should not try to do the impossible, it is advisable that the national parliaments investigate the issue.

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