The derivative is algebra

Improving the logical base of calculus on the issue of 'division by zero'

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Abstract

Calculus can be developed with algebra and without the use of limits and infinitesimals. Define y / x as the 'outcome' of division and $y // x$ as the 'procedure' of division. Using $y // x$ with x possibly becoming zero will not be paradoxical when the paradoxical part has first been eliminated by algebraic simplication. The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and its Cauchy shorthand for the derivative $\lim(\Delta x \to 0)\, \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken. Instead, using $\Delta f // \Delta x$ and then setting $\Delta x = 0$ is not paradoxical at all. Much of calculus might well do without the limit idea and it could be advantageous to see calculus as part of algebra rather than a separate subject. This is not just a didactic observation but an essential refoundation of calculus. E.g. the derivative of $|\,x\,|$ traditionally is undefined at $x = 0$ but would algebraically be $\mathrm{sgn}[x]$, and so on.

Introduction

The zero has been giving trouble, before its invention and afterwards. Mathematics solved the paradoxes of the division by zero by forbidding it. But the problem persisted in calculus, where the differential quotient relies on infinitesimals that magically are both non-zero before division but zero after it. Karl Weierstraß (1815-1897) is credited with formulating the strict concept of the limit to deal with the differential quotient.

Consider the following expressions, three well-known and the fourth a new design.

(1) The difference quotient $\Delta f / \Delta x = (f[x + \Delta x] - f[x]) / \Delta x$ for $\Delta x \neq 0$. Note that one would see this as a result and not as a procedure.

(2) The differential quotient or derivative $f[x] = \mathbf{df} / \mathbf{dx} = \lim(\Delta x \to 0)\, \Delta f / \Delta x$.

(3) The current theoretical true meaning of the derivative with outcome value *L:* $\forall \varepsilon > 0 \;\; \exists \delta > 0$ so that for $0 < |\,\Delta x\,| < \delta$ we have $|\,\Delta f / \Delta x - L\,| < \varepsilon$.

(4) The new suggestion: $f[x] = \mathbf{df} / \mathbf{dx} = \{\Delta f // \Delta x,\ \text{then set } \Delta x = 0\}$. This // means first simplifying the difference quotient and then extending the domain with $\Delta x = 0$. The next step in the procedure subsequently sets Δx to zero.

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Let us consider the various properties. To do so, let us first develop the notion of dynamic division (//). But first of all a short history will given a useful overview.

History of the decimal system and the zero

The West often speaks about Arabic digits but according to Van der Waerden (1975:58) the Arabs speak about Indian digits so we better follow them:

> "Our digits derive from the Gobar digits which were used in Moorish Spain. The East-Arabic digits are still in use to-day in Turkey, Arabia and Egypt; they are called "Indian digits". It is clear that both were derived from the Brahmi-digits."

Barrow (1993:85) mentions that the Babylonians of 300-200 BC already had a symbol to indicate a blank spot. Van der Waerden refers to Freudenthal 1946 for the most likely story on what happened next. It can be observed that Ptolemy in 150 AD wrote whole numbers with Roman numerals but fractions sexagesimally following the Babylonians – and in this positional system he wrote "o" for "ουδεν" ("nothing") when a position was blank. Apparently the Hindus became familiar with Greek astronomy from 200 AD onwards. The Hindus already had a decimal positional system though of some complexity. They used rhymes and verses to remember long numerical tables, but blank places apparently broke the rhythm and it would have come as an idea that those places could be filled with sounds too.

Van der Waerden (1975:55):

> "Bhaskara I, a pupil of Aryabhata, introduced an improved system, which is positional and has zero; it has the further advantage of leaving the poet greater freedom in the choice of syllables and thus enabling him better to meet metrical requirements. According to Datta and Singh, this Bhaskara lived around 520. Like Aryabhata, he begins with the units, followed by the tens, etc., (…) The first to reverse the order (as far as we know) was Jinabhadra Gani, who lived about 537, according to Datta and Singh."

Van der Waerden (1975:57) summarizes:

> "Along with Greek astronomy, the Hindus became acquainted with the sexagesimal system and the zero. They amalgamated this positional system with their own; to their own Brahmin digits 1 – 9, they adjoined the Greek o and they adopted the Greek-Babylonian order. It is quite possible that things went in this way. This detracts in no way from the honor due to the Hindus; it is they who developed the most perfect notation for numbers, known to us."

Clearly, when the zero arrived in Europe again via the Moors in Spain, it helped that astronomers were already used to it. The impact however came from the package deal with the decimal notation in general, that appeared very useful in commerce, see also Struik (1977).

Verbs versus nouns, for division

Western mathematics thus had to wait till 1200 AD before the zero came from India via Arabia together with the Indian digits – where both "zero" and "cipher" are jointly derived from the Arabic "sifr" = "empty". Indian numerals are easier to work with than Roman numerals, e.g. try to divide MCM by VII, yet this advance came with the cost

that the zero caused a lot of paradoxes. Western math solved most problems by forbidding division by zero. However, we might also try some algebra.

Dijksterhuis (1990) suggests that the ancient Greeks did not develop algebra – and subsequently analytical geometry – since they used their alphabet to denote numbers. Thus $\alpha + \alpha = \beta$ already had the meaning $1 + 1 = 2$, whence it would be less easy to hit upon the idea to use α as a variable. We too would consider it strange to use e.g. 15 as a variable ranging over $-\infty$ to $+\infty$. This explanation is not entirely convincing since the Greeks did use names like "Plato" or "Aristotle" and thus might have used a name to denote a variable – like "Variabotle" – though this then should not be a number again. There is also a subtle difference between a variable that is thought to already directly stand for an (unknown) number and a variable that is only a place holder or label that later is assigned a number. Notation clearly was one of the obstacles to overcome. Let us now assume that we are familiar with algebra and that someone announces the new invention of the zero.

Let us distinguish the passive division result from the active division process. In the active mode of dividing y by x we may first simplify algebraically under the assumption that $x \neq 0$ while subsequently the result can also be declared valid for $x = 0$. This means extending the domain, and not setting $x = 0$.

There is already an active notion (verb) in taking a ratio $y : x$. But a ratio is not defined as the above, for $x = 0$. Mathematicians will tend to regard y / x as already defined for the passive result (noun) without simplification – i.e. defined except for $x = 0$. Thus the intended other active notion would be new. Denote it as $y \mathbin{//} x$. Others who aren't professional mathematicians, in particular pupils or students, will tend to take y / x as an active process and they might denote $y \mathbin{//} x$ for the passive result. All in all, it would not matter much, since we might continue to write y / x and allow both interpretations depending upon context. In that way the paradoxes of division by zero are actually explained, i.e. by confusion of approach or perspective.

Definition

To make this strict, let y / x be as it is used currently by mathematicians and $y \mathbin{//} x$ be the following process or program:

> $y \mathbin{//} x \equiv \{\ y / x$, unless x is a variable and then: assume $x \neq 0$, simplify the expression y / x, declare the result valid also for the domain extension $x = 0\ \}$.

Thus simplification only holds for variables but not for numbers. Thus $x \mathbin{//} x = 1$ but $4 \mathbin{//} 0$ generates $4 / 0$ which is undefined. x / x is standard undefined for $x = 0$.

General application

There is no need to be very strict about always writing "//". Once the idea is clear, we might simply keep on writing "/". An expression like $(1 - x^2) / (1 - x)$ would be undefined at $x = 1$ but the natural tendency is to simplify to $1 + x$ and not to include a note that $x \neq 1$, since there is nothing in the context that suggests that we would need to be so pedantic, see **Table 1**. The current teaching and math exam practice is to use the division y / x as a hidden code that must be cracked to find where $x = 0$ but it should rather be the reverse, i.e. that such undefined points must be explicitly provided if those values are germane to the discussion. Standard graphical routines also skip the undefined point, see **Figure 1**, requiring us to give the special point if we really want a hole.

Table 1: Symplification and continuity

Traditional definition overload	With the dynamic quotient
$f(x) = (1 - x^2) / (1 - x) = 1 + x$ if $x \neq 1$ $f(1) = 2$	$(1 - x^2) \mathbin{//} (1 - x) = 1 + x$

Figure 1: Graph of $(1 - x^2) \mathbin{//} (1 - x)$
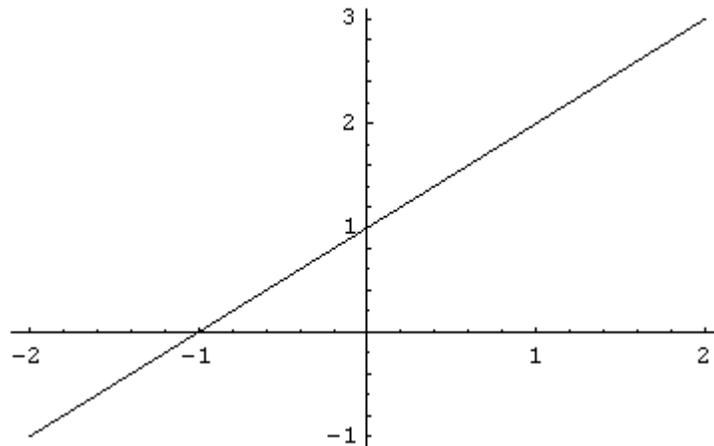
Subtleties

The classic example of the inappropriateness of division by zero is the equation $(x - x) (x + x) = x^2 - x^2 = (x - x) x$, where division by $(x - x)$ causes $x + x = x$ or $2 = 1$. This is also a good example for the clarification that the rule that we should never divide by zero actually means that we must distinguish between:

- creation of a fraction by the choice of the *infix* between $(x - x) (x + x)$ and $(x - x)$

- handling of a fraction such as $(x - x) (x + x)$ *infix* $(x - x)$ once it has been created.

The first can be the great sin that creates such nonsense as $2 = 1$, the second is only the application of the rules of algebra. In this case, $x - x = 0$ is a constant and not a variable, so that simplification generates a value Indeterminate for both / and //.

Also $a (x + x) / a$ would generate $2x$ for $a \neq 0$ and be undefined for $a = 0$. However, the expression $a (x + x) \mathbin{//} a$ gives $2x$, and this result would also hold for $a = 0$, even while it then is possible to write $a = x - x = 0$, since then it is an instant and not a variable.

Another conclusion is that calculus might use algebra and the dynamic quotient for the differential quotient instead of referring to infinitesimals or limits.

Requirements

Clearly, mathematics education already takes account of these kind of aspects in some fashion. In early excercises pupils are allowed to divide $2 \, a / a = 2$ without the definition overload. At a certain stage though the conditions are enforced more strictly. The topic

of discussion is not only that this stage can be a bit later but also that the transition can be smoother, also for the rest of the education, by the distinction between / and //. For the mathematically inclined pupils or students graduating at highschool one would require that they are aware that x / x is undefined for $x = 0$ and that they can find such points.

The old approaches on the derivatives

The theory of limits is problematic. The limit of e.g. x / x for $x \to 0$ is said to be defined for the value $x = 0$ on the horizontal axis yet not defined for actually setting $x = 0$ but only for x getting close to it, which is paradoxical since $x = 0$ would be the value we are interested in. Mathematicians get around this by defining a special function $f[x] = x / x$ with split domain but this requires a separate "f" and it is faster to write $x // x$.

Also, the interpretation given by Weierstraß can be rejected since that definition of the limit still excludes the value (at) $\Delta x = 0$ which actually is precisely the value of interest at the point where the limit is taken.

While the Weierstraß approach uses predicate logic to identify the limit values, the new alternative approach uses algebra, the logic of formula manipulation.

Leibniz, Newton, Cauchy and Weierstraß were trained to regard y / x as sacrosanct such that it indeed doesn't have a value for $x = 0$. They worked around that, so that algebraically y / x could be simplified before x got its value. While doing so, they created a new math that appeared useful for other realms. These new results gave them confidence that they were on the right track. Yet, they also created something overly complex and essentially inconsistent. Infinitesimals are curious constructs with no coherent meaning. Bishop Berkeley criticized the use of infinitesimals, that were both quantities and zero: who could accept all that, need, according to him, "not be squeamish about any point in divinity". The standard story is that Weierstraß set the record straight. However, Weierstraß's limit is undefined at precisely the relevant point of interest. 'Arbitrary close' is a curious notion for results that seem perfectly exact. When we look at the issue from this new algebraic angle, the problem in calculus has not been caused by the "infinitesimals" but by the confusion between "/" and "//".

The present discussion can be seen as reviving the Cauchy approach but providing another algebraic interpretation that avoids the use of 'infinitesimals'. The impetus comes from the notion of the dynamic quotient in algebra. We cannot change properties of functions but we can change some interpretations. Undoubtedly, the notion of the limit and Weierstraß's implementation remain useful for specific purposes. That said, the discussion can be simplified and pruned from paradoxes.

Struik (1977) incidently states that Lagrange already saw the derivative as algebraic. See there for details and why contemporaries thought his method unconvincing.

The algebraic approach

In a way, the new algebraic definition is nothing new since it merely codifies what people have been doing since Leibniz and Newton. In another respect, the approach is a bit different since the discussion of 'infinitesimals', i.e. the 'quantities vanishing to zero', is avoided.

The derivative deals with formulas too, and not just numbers. It uses both that $\Delta f // \Delta x$ extends the domain to $\Delta x = 0$ and that the instruction "set $\Delta x = 0$" subsequently restricts the result to that point.

Since we have been taught not to divide without writing down that the denominator ought to be nonzero, the following explanation will help for the proper interpretation of the derivative: first the expression is simplified for $\Delta x \neq 0$, then the result is declared valid also for the domain $\Delta x = 0$, and then Δx is set to the value 0. The reason for this declaration of validity resides in the algebraic nature of the elimination of a symbol, as in $x \mathbin{/\!/} x = 1$, and the algebraic considerations on 'form'.

The true problem is to show why this new definition of df / dx makes sense.

Stepwise explanation of the algebraic approach

Let us create calculus without depending upon infinitesimals or limits or division by zero.

(1) We distinguish cases $\Delta x \neq 0$ and $\Delta x = 0$, and the (*) implicit or (**) explicit definition of relative error $r[\Delta x]$.

(2) Let $F[x]$ be the surface under $y = f[x]$ till x, for known F and unknown f that is to be determined (note this order). For example $F[x] = x^2$ gives a surface under some f and we want to know that f.

(3) Then the change in surface is $\Delta F = F[x + \Delta x] - F[x]$. When $\Delta x = 0$ then $\Delta F = 0$.

(4) The surface change can be approximated in various ways. For example $\Delta F \approx \Delta x\, y = \Delta x\, f[x]$, or $\Delta F \approx \Delta x\, f[x + \Delta x]$, or inbetween with $\Delta y = f[x + \Delta x] - f[x]$, $\Delta F \approx \Delta x\ (y + \Delta y/2)$. Of these $\Delta F \approx \Delta x\, y$ is the simplest expression with explicit y.

(5) The error will be a function of Δx again. We can write ΔF in terms of $y = f[x]$ (to be found) and a general error term $\varepsilon[\Delta x]$, where the latter can also be written as $\varepsilon[\Delta x] = \Delta x\, r[\Delta x]$ where $r[\Delta x]$ is the relative error. When $\Delta x = 0$ and thus $\varepsilon[\Delta x] = 0$ then the relative error can be seen as undefined and it can be set to zero by definition.

(6) We have these relations where we multiply by zero and nowhere divide by zero or infinitesimals.

	(*) *Implicit definition of r*	(**) *Explicit definition of r*
$\Delta x \neq 0$	$\Delta F = y\, \Delta x + \varepsilon[\Delta x]$	$r[\Delta x] \equiv \Delta F / \Delta x - y$
$\Delta x = 0$	$\Delta F = 0 = c\, \Delta x + \varepsilon[\Delta x]$ for any c; select $c = y$	$r[\Delta x] \equiv 0 = c - y$ for $c = y$

(7) Simplify $\Delta F / \Delta x$ algebraically for $\Delta x \neq 0$ and determine whether setting $\Delta x = 0$ gives a defined outcome. When the latter is the case, take c as that outcome.

(8) Thus $c = \{\Delta F \mathbin{/\!/} \Delta x,\ \text{then set } \Delta x = 0\}$.

(9) We then find $c = y = f[x]$ which can be denoted as $F'[x]$ as well.

For example, the derivative for $F[x] = x^2$ gives $dF / dx = \{(x + \Delta x)^2 - x^2) \mathbin{/\!/} \Delta x,\ \text{then } \Delta x := 0\} = \{2x + \Delta x,\ \text{then } \Delta x := 0\} = 2x$. This contains a seeming 'division by zero' while actually there is no such division.

The selection of $c = y$ is based upon 'formal identity'. This is a sense of consistency or 'continuity', not in the sense of limits but in the sense of 'same formula', in that (*) and (**) have the same form (each seen per column) irrespective of the value of Δx.

The deeper reason (or 'trick') why this construction works is that (*) evades the question what the outcome of $\varepsilon[\Delta x]\ //\ \Delta x$ would be but (**) provides a definition when the error is seen as a formula. Thus, (*) and (**) give exactly what we need for both a good expression of the error and subsequently the 'derivative' at $\Delta x = 0$. The deepest reason (or 'magic') why this works is that we have defined $F[x]$ as the surface (or integral), with both (a) an approximation and (b) an error for *any* approximation that still is accurate for $\Delta x = 0$. When the error is zero then we know that $F[x]$ gives the surface under the $c = y = f[x] = F'[x]$ which is the function that we found.

In summary: The program is $F'[x] = dF\ /\ dx \Leftrightarrow \{\Delta F\ //\ \Delta x,$ then set $\Delta x = 0\}$. The definitions (*) and (**) give the rationale for extending the domain with $\Delta x = 0$, namely form.

Implications

Perhaps other approaches can be found in the same manner. In the mean time it seems that the proper introduction to calculus is to start with a function that describes a surface and then find the derivative. Since we only use equivalences, this also establishes that the reverse operation on the derivative gives a function for the surface.

The relation to the slope only arises in point (4) above. Traditionally the derivative is created from the question to find the slope at some point of a function. This tradition also suggests a separate development for the integral, e.g. with Riemann sums. Instead, here we find that the slope comes as a fast corollary – seeing that $\Delta F\ //\ \Delta x$ would be the tangent if it is defined. The original choice however derives from 'simplicity'. Clearly, this is not a sharply defined notion. The selection of this approximation might also be seen as derived for its corollary – the interpretation as the slope. Be that as it may, there still is a difference in starting out with determining the surface versus the slope.

Let us look closer into the difference between starting from slopes or from surfaces.

The derivative of $|x|$ is traditionally undefined at $x = 0$ but would algebraically become sgn$[x]$. For $x \neq 0$, we can consider the various combinations and find the normal result, sgn$[x]$. For $x = 0$ the dynamic quotient gives $(|x + \Delta x| - |x|)\ //\ \Delta x = |\Delta x|\ //\ \Delta x = $ sgn$[\Delta x]$. Setting $\Delta x = 0$ gives 0. Hence in general $|x|' = $ sgn$[x]$.

The traditional approach to $|x|$ is a bit complicated. Cauchy naturally gives 0 at 0 too. However, there is a multitude of 'tangent' lines at 0, that is, when tangency is not defined as having the same slope as the function (which slope is undefined at 0) but as having a point in common that is no intersection. Traditionally the derivative is used for finding slopes and then the amendment on Cauchy was to hold that the right derivative differs from the left derivative, hence traditionally there is no general derivative.

In our approach, when we are interested in slopes, then it remains proper to consider these left and right derivatives. However, better terms are derivatives "to the left" and "to the right". We do not need to speak about limits but merely can point to the different values of the derivative sgn$[x]$ in the intervals $(-\infty, 0)$, $[0]$, $(0, +\infty)$. Depending upon the definition of 'tangent': (a) "Tangent" lines that have the point $\{0, 0\}$ in common without intersection then can have slopes from -1 to 1. (b) "Tangent" lines that have the same slope as the function however have only the three slopes $-1, 0, 1$.

The dynamic quotient is the leading impetus here and the issue starts with algebra so that slopes come in only second. $|x|$ is the surface under some function f. Any approximation of changes in the surface, when the surface value is $|0| = 0$, finds a perfect answer with

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zero relative error by requiring $f[0] = 0$. The general function appears to be sgn[x]. The choice to extend the domain of Δx with value 0 at $x = 0$ derives from a notion of consistency of the *form* of the relative error in the approximation. This is sufficient though not necessary. One could argue that the relative error is not defined when $\Delta x = 0$ but this runs counter to our choice to define it as 0. This choice again relates to the form of the relations in step (6).

Students

Generations of students have been suffering. Teachers of math seem to have overcome their own difficulties and thereafter don't seem to notice the inherent vagueness.

Students not only suffer from the vagueness but also from the notation. Many forget to write "lim($\Delta x \to 0$)" as the first part of each differential quotient, each separate line again and again for each step of the deduction, assuming that stating it once should be sufficient to express that they are taking the limit. Some 'take the limit' so that for them Δx has become 0, and then, just to be sure, they still mention "… $+ \Delta x$" arguing that it should not matter when you add 0. Those 'official mathematical errors' will be past.

Conversely, if the new notation of dynamic division is adopted also for general purposes then the algebraic origin of the derivative will be sooner recognized, strengthening the insights in logic and algebra. Time can be won for more relevant issues.

Teachers may be less tempted to distinguish between 'those who know the truth' (Deep Calculus, the ε and δ) (who thus actually are wrongfooted) and 'those who only learn the tricks' (Superficial Calculus).

Didactics remain an issue. Above nine steps are somewhat elaborate while the short program $\{\Delta F // \Delta x$, then set $\Delta x = 0\}$ sums it up and suffices. Possibly some randomized controlled trials in education would bring more light in the question what explanation works where.

The chain rule

The chain rule is an important result and can found directly as follows.

$$df / dx = \{\Delta f // \Delta x, \text{ then set } \Delta x = 0\}$$

$$= \{\Delta f // \Delta g * \Delta g // \Delta x \text{ for } (\Delta x = 0 \Leftrightarrow \Delta g = 0), \text{ then set } \Delta x = 0 \}$$

$$= \{\Delta f // \Delta g, \text{ then set } \Delta g = 0\} * \{\Delta g // \Delta x, \text{ then set } \Delta x = 0\}$$

$$= df / dg * dg / dx$$

The derivative of an exponential function

The derivative of an exponential function follows from the chain rule and the presumption that Exp[x] = e^x is the fixed point in differentiation:

$$\frac{\partial a^x}{\partial x} = \frac{\partial e^{x\,\text{Rex}[a]}}{\partial x} = e^{x\,\text{Rex}[a]} . \text{Rex}[a] = a^x \, \text{Rex}[a]$$

Where Rex[a] = Ln[a] is the "recovered exponent" function. The reasoning thus is:

(i)   All functions can be expressed as an exponential function for any nonnegative base number b, as $f[x]$ = Exp[b, Rex[b, $f[x]$]].

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(ii)   We presume that in this class of all possible bases there is a fixed point in differentiation. Call this base the number e. Thus by definition $(e^x)' = e^x$.

(iii)   We can calculate e from the property $e^x \equiv \boldsymbol{d}\, e^x / \boldsymbol{dx} = \{e^x\,(e^h - 1)\ //\ h,\ \text{set}\ h = 0\}$. This gives $1 = \{(e^h - 1)\ //\ h,\ \text{set}\ h = 0\}$. By setting $(e^h - 1) = h$ and solving $e = (1 + h)^\wedge(1/h)$ we find the approximate value of e by taking h close to zero.

(iv)   That there is an actual number e with 'infinite accuracy' follows from (iii) and from notions of continuity ('there are no holes between 2 and 3').

(v)   From the chain rule we find in general $\text{Rex}[a] = \{(a^h - 1)\ //\ h,\ \text{set}\ h = 0\}$.

Thus, the dynamic quotient $(a^h - 1)\ //\ h = (e^{h\ \text{Rex}[a]} - 1)\ //\ h$ does not simplify easily. However, when we use the chain rule then we can avoid using this explicit expression and actually find its value by implication.

Some meta-comments are:

(a) The number e remains an algebraic concept like the number π.

(b) The procedure to first presume e and its property, and only then calculate / approximate it, and thus prove its existence by calculation, summarizes an intricate historical development, but does not invalidate the existence proof.

(c) In this case approximate values for e are found as we would normally take a limit. But the limit is not applied for the derivative.

(d) The notion of a limit by itself still has its advantages, e.g. for the limit to infinity, and thus for $1\ //\ 0$ again. It would not be right not to mention limits in education.

(e) There remains a distinction however between algebraic simplification and extension of the domain on the one hand and the traditional concept of a limit on the other hand. This distinction causes the insight that the derivative is an algebraic notion rather than dependent upon infinitisemals.

(f) Given that limits can be defined in acceptable manner suggests that calculus can be developed by using limits. Indeed, complex ways can be used for what is simple.

Conclusion

History is a big subject and we should be careful about drawing big historical lines. But the following seems an acceptable summary of the situation where we currently find us after the historical introduction of the zero.

Historically, the introduction of the zero in Europe around AD 1200 gave so many problems that once those were getting solved, those solutions, such as that one cannot divide by zero, were codified in stone, and pupils in the schools of Europe would meet with bad grades, severe punishment and infamy if they would sin against those sacrosanct rules. Tragically, a bit later on the historical timeline, division by zero seemed to be important for the differential quotient. Rather than reconsidering what 'division' actually meant, and slightly modifying our concept of division, Leibniz, Newton, Cauchy and Weierstraß decided to work around this, creating the concepts of infinitesimals or the limit. In this way they actually complicated the issue and created paradoxes of their own.

The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and the derivative's shorthand $\lim(\Delta x \to 0)\ \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken.

Logical clarity and soundness can be restored by distinguishing between the (formal) act of division and the (numerical) result of division. Using $\Delta f \;//\; \Delta x$ and then enlarging the domain and setting $\Delta x = 0$ is not paradoxical at all.

The distinction between static and dynamic division suggests that the Weierstraß purity may be overly pedantic for the main body of calculus. The exact definition of the limit is of great value but not necessarily for all of calculus. Indeed, 'most' derivatives can be found without the Weierstraß technical purity and 'many' courses already teach calculus without developing that purity. Thus there is ample cause to bring theory and practice more in line.

Literature

PM. Colignatus is the name of Thomas Cool in science. See http://www.dataweb.nl/~cool.

PM. The present discussing is taken from my new book "Elegance with substance" (2009) and improves upon earlier statements. When retyping Colignatus (2007a) "A logic of exceptions" (ALOE) I got inspired to also consider the paradoxes caused by zero and included a page with the approach discussed here. The discussion by Gill (2008) of ALOE does not discuss this. Colignatus (2007b) is an elaboration of that page in ALOE. The page in ALOE can remain as it is but 2007b would be superseded by this discussion.

Barrow, J. (1993), "Pi in the sky. Counting, thinking and being", Penguin
Colignatus (2007a), "A logic of exceptions" (ALOE), see
    http://www.dataweb.nl/~cool/Papers/ALOE/Index.html
Colignatus (2007b), "Improving the logical base of calculus on the issue of "division by
    zero", http://www.dataweb.nl/~cool/Papers/ALOE/2007-07-31-
    ImprovingTheLogicOfCalculus.pdf
Colignatus (2009), "Elegance with substance",
    http://www.dataweb.nl/~cool/Papers/Math/Index.html
Dijksterhuis, H. (1990), "Clio's stiefkind. Een keuze uit zijn werk door K. van Berkel", Bert
    Bakker
Gill, R.D. (2008), "Book reviews. Thomas Colignatus. A Logic of Exceptions: Using the
    Economics Pack Applications of *Mathematica* for Elementary Logic", NAW 5/9 nr. 3
    sept., http://www.math.leidenuniv.nl/~naw/serie5/deel09/sep2008/reviewssep08.pdf
Scheelbeek, P.A.J. and W. Verdenius (1973), "Analyse", syllabus Mathematical Institute,
    Rijksuniversiteit Groningen
Struik, D. (1977), "Geschiedenis van de wiskunde", SUA. Translated from "Concise history
    of mathematics", Dover 1948
Tall, D. (2002), "Three Worlds of Mathematics", Bogota, Columbia, July 2–5, 2002, see
    http://www.warwick.ac.uk/staff/David.Tall/themes/procepts.html
Waerden, B.L. van (1975), "Science Awakening I", 4th edition, Kluwer
Waerden, B.L. van (1983), "Geometry and algebra in ancient civilizations", Springer