Foundations of Mathematics. A Neoclassical Approach to Infinity

Pro Occam Contra Cantor.
Proper Constructivism with Abstraction.
A condition by Paul of Venice (1369-1429) solves Russell's paradox, blocks Cantor's diagonal argument, and provides a challenge to ZFC.
Two results on ZFC: (1) If ZFC is consistent then it is deductively incomplete, (2) ZFC is inconsistent.

Companion to

A Logic of Exceptions (1981, 2007, 2011)

and

Elegance with Substance (2009, 2015)

Thomas Colignatus

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Colignatus is the science name of Thomas Cool (1954), econometrician (Groningen 1982) and teacher of mathematics (Leiden 2008). He worked at the Dutch Central Planning Bureau (CPB) in 1982-1991. His analysis on unemployment met with censorship by the CPB Directorate and he was dismissed with an abuse of power. He advises to a boycott of Holland till this censorship of science is resolved. He is candidate for President of the European Union for the Dutch Sociaal Liberaal Forum.

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NUR
918 Wiskunde algemeen
921 Fundamentele wiskunde
846 Didactiek

MSC2010
In bold the three major ones if required to select
General
00A30 Philosophy of mathematics
Mathematics education research
00A35 Methodology of mathematics, didactics
97D20 Mathematics Education - Philosophical and theoretical contributions (maths didactics)
97E60 Mathematics Education - Foundations: Sets, relations, set theory
97C70 Mathematics Education - Research: Teaching-learning processes
97B10 Mathematics Education - Educational research and planning
Mathematics itself
03B50 Many-valued logic
03E30 Axiomatics of classical set theory and its fragments
03E35 Consistency and independence results
03E70 Nonclassical and second-order set theories
03F65 Other constructive mathematics

Preface

The target readership are (1) students with an interest in methodology of science and the foundations of mathematics – for example students in physics, engineering, economics, psychology, thus a broad group that uses mathematics and not only those majoring in mathematics – and (2) fellow teachers of mathematics who are sympathetic to the idea of bringing set theory and number theory into general mathematics education – while avoiding the *New Math* disaster in the 1960s in highschool. [1]

Readers would be interested in:

(A) Constructivism with Abstraction, as a scientific methodology
(B) Particulars about infinity and number theory, within foundations and set theory
(C) Correction of errors within mathematics on (B) caused by neglect of (A).

Other readers are (3) research mathematicians, but while they would benefit from the last correction in (C), they must mend for that they are not in the prime target groups. They would start with pages 61-72 below (Colignatus (2015g)) and then restart here again.

Set theory and number theory would be crucial for a better educational programme:

(i) They greatly enhance competence and confidence
(ii) They open up the mind to logical structure and calculation also in other subjects
(iii) They are fundamental for learning and teaching themselves.

The world can be amazed that (A) and (B) are not taught systematically in current school and first year higher education. There are two explanations. One is the mentioned *New Math* disaster in the 1960s. Another more hidden cause are the *transfinites* created by Georg Cantor (1845-1918). When a mathematics teacher starts on the topics of numbers and set theory, and then infinity, then he or she feels obliged to discuss these transfinites. However, her or she also feels doubt whether these should be taught. For highschool and first year students they might be too complex and paradoxical. People in real life have no application for these transfinites and it makes little sense to have transfinites in the highschool diploma. They are relevant purely for mathematicians – and for a particular branch of mathematics as well. Thus, mathematics teaching is stuck. A mathematical *curl* causes so much complexity and irrelevance *that the wonderful basics are not taught.* This book proposes to cut the knot. It adds the bitter irony that Cantor's analysis appears to be misguided. Neglect of (A) made generations of mathematicians blind to some crucial errors.

☙

As a student in 1980 I greatly benefitted from the admirable book by Howard DeLong (1971) *A profile of mathematical logic.* His book provides the mixture of history, philosophy and mathematics that I still find the best approach. I did not agree with some deductions though, and my response was *A Logic of Exceptions* (ALOE) (1981 unpublished, 2007, 2011). Originally I focused on the Liar paradox and accepted the transfinites. In 2007 I however saw how Cantor's proof on the power set linked up to Russell's paradox. In 2009 I collected my observations on mathematics education in *Elegance with Substance* (EWS). Combining both issues caused two longer papers – in the next paragraph – that comprise this book. An advice to readers is to indeed look at ALOE, EWS and DeLong (1971) too.

☙

This book contains some original mathematics. This only serves the main purpose of this book. The original papers have abbreviations CCPO-PCWA and PV-RP-CDA-ZFC. Their arguments have been polished up so that this book replaces them. Discussions have been cut up, re-edited and dispersed over chapters, so that they actually create this book. The papers have a *Contra Cantor* (CC) flavour but the book is constructively *Pro Occam* (PO).

[1] https://en.wikipedia.org/wiki/New_Math

Abstract

Contra Cantor Pro Occam - Proper Constructivism with Abstraction

> Context • In the philosophy of mathematics there is the distinction between *platonism* (realism), *formalism*, and *constructivism*. There seems to be no distinguishing or decisive experiment to determine which approach is best according to non-trivial and self-evident criteria. As an alternative approach it is suggested here that philosophy finds a sounding board in the *didactics of mathematics* rather than mathematics itself. Philosophers can go astray when they don't realise the distinction between mathematics (possibly pure modeling) and the didactics of mathematics (an empirical science). The approach also requires that the didactics of mathematics is cleansed of its current errors. Mathematicians are trained for abstract thought but in class they meet with real world students. Traditional mathematicians resolve their cognitive dissonance by relying on tradition. That tradition however is not targetted at didactic clarity and empirical relevance with respect to psychology. The mathematical curriculum is a mess. Mathematical education requires a (constructivist) re-engineering. Better mathematical concepts will also be crucial in other areas, such as e.g. brain research. **> Problem** • Aristotle distinguished between potential and actual infinite, Cantor proposed the transfinites, and Occam would want to reject those transfinites if they aren't really necessary. My book *"A Logic of Exceptions"* already refuted 'the' general proof of Cantor's Conjecture on the power set, so that the latter holds only for finite sets but not for 'any' set. There still remains Cantor's diagonal argument on the real numbers. **> Results** • There is a *bijection by abstraction* between $\mathbb{N}$ and $\mathbb{R}$. Potential and actual infinity are two faces of the same coin. Potential infinity associates with counting, actual infinity with the continuum, but they would be 'equally large'. The notion of a limit in $\mathbb{R}$ cannot be defined independently from the construction of $\mathbb{R}$ itself. Occam's razor eliminates Cantor's transfinites. **> Constructivist content** • Constructive steps S_1, ..., S_5 are identified while S_6 gives non-constructivism (possibly the transfinites). Here S_3 gives potential infinity and S_4 actual infinity. The latter is taken as 'proper constructivism with abstraction'. The confusions about S_6 derive rather from logic than from infinity.

ZFC is inconsistent. A condition by Paul of Venice (1369-1429) solves Russell's paradox, blocks Cantor's diagonal argument, and provides a challenge to ZFC

Paul of Venice (1369-1429) provides a consistency condition that resolves Russell's Paradox in naive set theory without using a Theory of Types. It allows a set of all sets. It also blocks the (diagonal) general proof of Cantor's Conjecture (in Russell's form, for the power set). The Zermelo-Fraenkel-Axiom-of-Choice (ZFC) axioms for set theory appear to be inconsistent. They are still too lax on the notion of a well-defined set. The transfinites of ZFC may be a mirage, and a consequence of still imperfect axiomatics in ZFC for the foundations of set theory. For amendment of ZFC two alternatives are mentioned: ZFC-PV (amendment of de Axiom of Separation) or BST (Basic Set Theory).

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Contents

List of theorems and conjectures

List of main symbols and abbreviations

$\{a, ..., b\}$	the set with elements *a* to *b*
Ø	the empty set, the set with no members, p 61
$\{x \mid p\}$	the set of elements *x* for which property *p* = *p*[*x*] applies
$\{x \mid p \uparrow q\}$	$\{x \mid p[x]$ unless $(p[x] \wedge q[x])$ is contradictory (also formally)$\}$, p 80
$\mathbb{N}[n]$	{0, 1, 2, ..., *n*}, a list of some natural numbers (cardinal), potential infinity
$\mathbb{N}$	{0, 1, 2, ...} or the natural numbers (cardinal), actual infinity, p 22
@	a step of abstraction, e.g. $\mathbb{N}[n]$ @ $\mathbb{N}$, p 23
$\mathbb{O}$	the ordinal numbers: (1^{st}, 2^{nd}, 3^{rd}, ...}, p 33
𝕤	{1, 2, ...} or a pure sequence created by the successor function, p 33
$\mathbb{S}$	set of all sets, p 55
$\mathbb{R}[d]$	numbers in [0, 1] with *d* the number of decimal digits, potential infinity
$\mathbb{R}$	the real numbers, actual infinity, p 42
x^H	1 / *x*, with *H* = -1 the Harremoës operator, pronounced as "eta", p 132
$S_1, ..., S_6$	steps (degrees) in constructivism, p 39
$x \in A$	*x* is an element of set *A*
$x \notin A$	*x* is not an element of set *A*
$\forall x$	for all *x*
$\exists x$	there is a *x* , or, for some *x*
$A \cup B$	union of set *A* with set *B*
$A \setminus B$	the difference: set *A* excluding the elements of *B*
$A \subseteq B$	*A* is a subset of *B,* possibly equal to *B*
$P[A]$	the power set of set *A,* i.e. the set of all subsets of *A,* see p 61
$A \sim B$	two sets *A* and *B* are equally large, with a one-to-one relation, p 32
$f: D \to R$	function *f* from domain *D* to range *R*: Any $x \in D$ gives at most one $f[x] \in R$
Б	bijection by abstraction between $\mathbb{N}$ and $\mathbb{R}$, so that $\mathbb{N} \sim \mathbb{R}$, p 43
$\neg p$	not-*p* (for proposition *p*)
† *p*	*p* is nonsensical, or, *not-at-all p,* see p 15 (dagger)
$p \Rightarrow q$	*p* implies *q,* or, if *p* then *q*
$p \Leftrightarrow q$	*p* is equivalent to *q,* or, *p* if and only if *q,* or, *p* iff *q*
$p \vee q$	*p* or q
$p \wedge q$	*p* and *q*
p & *q*	*p* and *q* (another notation)
∞	infinity, ambiguously potential or actual, also "undefined"
$[a, b]$	closed interval $a \le x \le b$
(a, b)	open interval $a < x < b$
ZFC	Zermelo-Fraenkel-Axiom-of-Choice system of axioms for set theory, p 61
ZFC-PV	ZFC adjusted for the consistency condition inspired by Paul of Venice, p 89
BST	Basic Set Theory, closer to naive set theory but with a PV condition, p 89

Part 1. Introduction

A neoclassical approach in mathematics

Two key rules for paradoxical self-reference

Our subject originally is set theory and number theory. Immediately relevant are Cantor's conjectures on the infinite. Subsequently, infinity becomes our main topic, and the earlier subjects are subsidiary. Cantor's conjectures rely on self-reference, which is a logical rather than a mathematical issue. Cases of self-reference that cause a contradiction are notoriously confusing. Two rules appear to be key in tackling such cases:

(1) We can maintain clarity by holding on to the notion of freedom of definition. When a restriction on this freedom generates a consistent framework, while release of the restriction generates confusion, then the restriction is to be preferred.

(2) There is a remarkable distinction between *not-well-defined* and *non-existent.* We can meaningfully discuss the existence or non-existence of something when we know what we are speaking about. When a rhinoceros exists, we can say whether it is in the room or not. For well-defined topics we can accept $p \lor \neg p$, known as *Tertium non datur* (TND). (The term *Law of the Excluded Middle* (LEM) needlessly imposes order.) But it may be that we are dealing with nonsense, $\dagger p$, so that in general only $p \lor \neg p \lor \dagger p$. For nonsense we may say *that it doesn't exist* but we actually mean to say *that the notion isn't well-defined*. Thus:

> In a dilemma $p \lor \neg p$ the non-existence of the one horn implies the other, but when there is nonsense or $p \lor \neg p \lor \dagger p$ then the non-existence of one horn cannot be turned into positive evidence for the other horn.

Constructivism

Hodges (1998) discusses submissions to the *Bulletin of Symbolic Logic* that claimed to refute Cantor but that failed on basic academic standards. This is indeed an area where intuition meets hard proof. Hodges sent me an email (August 10 2012) that he allows me to quote from:

> "You are coming at Cantor's proof from a constructivist point of view. That's something that I didn't consider in my paper, because all of the critics that I was reviewing there seemed to be attacking Cantor from the point of view of classical mathematics; I don't think they knew about constructivist approaches. Since then some other people have written to me with constructivist criticisms of Cantor. There is not much I can say in general about this kind of approach, because constructivist mathematicians don't always agree with each other about what is constructivist and what isn't."

The core of this book is the new definition of *bijection by abstraction.* [2] This new definition should appeal to all those who have had intuitive misgivings about Cantor's proof. The definition includes an aspect of *completion* that some readers may consider rather classical and non-constructivist. This book also discusses where Cantor's proof goes wrong. I suppose that there will be discussion about this but consider this of secondary value. It is more important to improve the didactics in highschool and matricola.

What is a neoclassical approach in mathematics ?

This book distinguishes:

- classical mathematics, from perhaps Pythagoras to Georg Cantor (1845-1918)
- traditional mathematics, from Cantor to *hopefully soon to end*
- neoclassical mathematics, as explained in this book.

[2] The notion of a bijection or a *one-to-one relationship* is defined on page 32.

The neoclassical approach in mathematics claims to maintain a better balance between *abstraction* and *empirics* – even though Cantor used the notion of abstraction himself. See Table 1 on page 28 for an overview of the differences. Below we will define abstraction, while we will assume that empirics are considered in the empirical sciences.

The core business of mathematics is abstraction, but abstraction by itself can lead people astray. Empirical researchers like engineers have more balance in their results by testing their ideas on nature. Mathematics lacks this counterveiling force of nature. The suggestion is to take the *education in mathematics* as the empirical area of relevance for mathematics. This suggestion also holds for philosophy in general, that also is in danger of getting lost in abstraction.

This book has a constructive and destructive component:

- It is constructive in two ways. It intends to help build up a better balance in doing and teaching mathematics. It also follows the philosophies of *nominalism* and *constructivism* as opposed to *realism* and *platonism* – see page 31. A key point is the *combination of constructivism with abstraction.* Abstraction might cause methods that some people may not regard as constructivist.
- The book is somewhat destructive is exposing the errors of traditional mathematics. It is a good question whether this destructive component is really so useful. Why not present the new approach and simply forget about old ways ? The main reason is that we are still too close to tradition, so that it is hard to let go. This book spends a major part of its attention to the errors of the traditional ways, to explain that these are errors, that there is a need for change, in particular w.r.t. the fundamental attitude that causes those errors.

While this book proposes a change in the way of doing mathematics, this is actually rather presumptuous since the author has little or any experience in research mathematics (RM). **Appendix D** contains a background. As an econometrician and teacher of mathematics – trying to reform school mathematics (SM) and matricola – my experience is that I meet with too many errors coming from research mathematicians: hence there must be something wrong at that source.

This introduction perhaps should also provide a definition of infinity, but we will do so later on. Before we proceed, it is useful to consider some paradoxes that can arise when you lose your sense of reality. Such paradoxes have led traditional mathematicians seriously astray on their notion of infinity.

Some paradoxes

(1) Consider the logic: I fit in my coat. My coat fits in my bag. Thus I fit in my bag.

A mathematician may be perfectly happy with this since the propositions are abstract and need not concern a real world and might only concern some topology. For an engineer, interested in an application to reality, the reasoning gives a problem. The assumptions seem true, the reasoning is sound, the conclusion is false, hence something is amiss.

The correction is straightforward: If I wear it, I fit in my coat. If nobody wears it, the coat fits in my bag. Conclusion: If I want to put the coat into the bag then I have to take it off.

(2) Axiomatics may create (seemingly) consistent systems that don't fit an intended interpretation. Van Bendegem (2012:143) gives the example that (a) 1 is small, (b) for each n, if n is small then n+1 is small, (c) hence all n are small. The quick fix is to hold that "small" can be nonsensical when taken absolutely, and that (a') 1 is less than 100, (b') for each n, if n is less than 100 n, then n+1 is less than 100 (n+1), (c') hence for all n, n is less than 100 n. The conclusion is that not all concepts or axiomatic developments are sensical in terms of the intended interpretation even though they may seem so.

(3) A *reductio ad absurdum* format of proof is as follows: one assumes hypotheses, deduces a contradiction, and concludes to the falsity of at least one of the hypotheses.

This format of proof seems to be a convenient way for the human mind to reason. This convenience may derive from cultural convention: there further doesn't seem to be anything special about it. The use of a contradiction may enhance confusion by nonsense though.

For example, define a *squircle* as a shape in Euclidean space that is both square and circular. A theorem is that it cannot exist. If it is square then the distance to the center will differ for corners and other points, and this contradicts the property of being circular. If it is circular, then it cannot have right angles, and this contradicts the property of being square. Hence a squircle does not exist in Euclidean space. QED. This is a fine proof.

Now suppose that this proof is not known. Consider the theorem that squares cannot exist in Euclidean space. We use the definition of squircles. There is a lemma that any square associates with a squircle, e.g. the squircle with a circle with the same area as the square. The proof then is: Take a square, find its associated squircle, and deduce a contradiction as done above. Square implies falsehood. Ergo, squares don't exist. QED.

We know that squares exist in Euclidean space, so something must be wrong. To pinpoint where it goes wrong may be less clear. After careful study we may conclude that the proof uses the *existence* of squircles as a hidden assumption. The lemma is false. Once this is spelled out, it is rather clear for this example.

We will see that it appears to be a bit more complex for Cantor's conjectures.

The difference between two-valued and three-valued logic is relevant here.

Consider the proof that squares don't exist. Let p = "Squares exist" and q = "Squircles exist." We had the lemma that $p \Rightarrow q$. Subsequently we find $p \Rightarrow \neg p$. Trivially $\neg p \Rightarrow \neg p$. The TND is that $p \vee \neg p$. Hence in all cases $\neg p$, or that squares do not exist.

With three-valued logic we must allow that there can be nonsense. Thus $p \vee \neg p \vee \dagger p$. What about the path $\dagger p \Rightarrow \neg p$? If we want squares to truly exist, the implication $\dagger p \Rightarrow \neg p$ must be false, and then we couldn't use $p \vee \neg p \vee \dagger p$ to conclude that $\neg p$. According to the truth-table (ALOE:183) an implication from *nonsense* is only true if the consequence is *true* or again *nonsense*. It is false when the consequence is *false*. Thus the path $\dagger p \Rightarrow \neg p$ is blocked when $\neg p$ is false. However, since an implication is false if the antecedens is true and the consequence is false, we must allow that $\dagger p$ is true, or that squares are a nonsensical idea. When we compare $\dagger p$ and $\dagger q$, which makes most sense to us ? A problem is: the definition of squircles and the lemma $p \Rightarrow q$ start to make the notion of a square nonsensical itself too. This however indicates that it is more likely that $\dagger q$ than $\dagger p$. In this simple case it is clear that the lemma $p \Rightarrow q$ *is* false. We rather look for a proof that $q \Rightarrow \neg q$. While three-valued logic is more complex than two-valued logic, it has the advantage stated in the rule on page 15, that the rejection of one horn is no proof for the other horn.

Structure of this book

This book has two main papers as its core. Some other short papers have been included that give relevant support. The development of the idea of some *neoclassical approach in mathematics* has been gradual. At some point there was the need to take stock, and to wonder what it all amounted to. Thus, this author did not sit down and decide to develop a new paradigm by deliberation. By consequence, the structure of this book may be somewhat less organised than one might expect from design. The book uses the material that is available, and reorganises it. The short papers could be included as they are, with a bit editing. The two main papers had such a complex argument that they were cut up and dispersed over the various parts and chapters, i.e. Colignatus (2012, 2013) (CCPO-PCWA) and (2014b, 2015) (PV-RP-CDA-ZFC).

The book has been divided into parts. Parts 1 and 2 are within naive set theory. Parts 3 and 4 are within formal ZFC (see below). Parts 5 and 6 contain discussions and more observations on constructivism. Some readers might prefer to look first at Part 6 on the philosophical aspects. For most readers it will be useful to begin by defining abstraction.