Reading Notes on “Conquest of the Plane” and “Contra Cantor Pro Occam”

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These notes result from discussing “*Conquest of the Plane*” (**COTP**) (2011b).

Relevant are “*A Logic of Exceptions*” (**ALOE**) (1981, 2007, 2011a) and “*Elegance with Substance*” (**EWS**) (2009). There is also the memo “*Brain research and mathematics education: some comments”* (2011c).

There are now:

- sheets http://thomascool.eu/Papers/COTP/2013-11-11-ColignatusStudiedagNVvW-English.pdf and
- a video (with apologies for the audio): http://youtu.be/gn_BKZaDa-o

(1) Main message

The main message is developed in EWS: that we would benefit from a parliamentary enquiry into mathematics education.

No such parliamentary enquiry would *seem* to be warranted if some points seem uncertain and still seem to require a mathematical discussion before they would be clarified. This would be a wrong conclusion. EWS and COTP provide a list of serious criticism on mathematics education. The point of interest for parliament is the list. A discussion on each point might generate “noise”. This however should not distract from the aggregate picture. The existence of the list itself causes an argument to restructure the industry of mathematics education and its research. I refer to EWS for the general case. Parliament cannot decide on content, but it can provide for an environment in which researchers look at the issues in scientific manner.

While there is no parliamentary enquiry yet, I might as well reply to some questions that some discussants have offered. These notes should help one to focus. (Call your representative.)

(2) Focus

ALOE, EWS, COTP & FMNAI should be evaluated for what they contribute, and not criticized for what they do not contribute.

My work relies on 99.999% of standard theory and changes 0.001%. COTP focusses on the didactics while leaving 'mathematics' intact. The didactics may change your perception of mathematics: but then we arrive in the academic discussion about what mathematics is. For example, you may measure angles on the base of 360º but now COTP changes the base to 1 turn: my suggestion is that this does not change the fundamental mathematical insight, but that the latter approach captures the insight in didactic better manner. The discussion then should be on that parliament should provide the funds to sharply determine, in a professional manner with proper experiments, what works best in education.

As professionals we should not doodle around with limited funds and perspective, as if this were a venture for amateurs.

The discussion should not be on personal views on what the 'right' mathematical approach should be. (Obviously, education should allow students to relate to history, and the use of 360º in history. But there is a distinction between understanding history and what method is best for education itself. It is fine if a student knows that Euclid writes about 360º but it is not fine if the student does not know what 360º is.)

In the discussion on COTP there quickly arises a question on completeness "why do you not explain X", where X is for example the theory of functions. Well, I don't include that because it is standard, and it would needlessly expand the number of pages without adding something new. COTP develops a structured path including the news and only includes standard stuff if it usefully enriches the story by providing perspective. If the difficulties of 360º cause us to delay on vectors then the proper choice gives us vectors. Please try to focus on the news and try to understand that. Please do not focus on how the story should be told if there would be no news.

ALOE, EWS, COTP & FMNAI offer insights and suggestions. Only those aspects are certain which are actually proven.

COTP is directed at didactics. The chapters on calculus state that it amounts to a refoundation of calculus. COTP does not claim that it develops and proves this. I only forward my impression that this appears to me to be the case. That is all. Those with less interest in didactics and more time for foundational analysis might be inspired to look into this. (PM. It is essential mathematics to look for counterexamples and develop new approaches to resolve those. Then it becomes an academic discussion what is the 'foundation': the original insight (that is reduced to a special case) or the reconstructed one (for a generalized case). When I suggest that a refoundation of calculus is involved I mean the first and not a potential later reconstruction.)

(3) History

COTP already acknowledges that it does not spend much attention to history.

These Reading Notes are partly a report about the road to repair that.

History shows a development from
(a) infinitesimals with Newton and Leibniz,
(b) algebra with Euler and Lagrange,
(c) numerical methods and limits with Cauchy and Weierstraß. (See Colignatus (2016c).)

Infinitesimals are revived in non-standard analysis. Lagrange's analytical approach is revived in the work by Schremmer & Schremmer. Both show better education results than the use of limits. Alain Schremmer indeed asks: Newton, Leibniz, Euler and Lagrange were already happy, why should we force more on students (if it is not really necessary) ? See COTP where COTP differs from all approaches.

(4) Infinite and infinitesimal

Limits are useful for mathematics, also in highschool. The point of COTP is that limits better should not be combined with the derivative, since this needlessly combines two learning goals. Mathematically, such combination is not necessary.

Colignatus (2016c) is a new statement on the relation to continuity.

See FMNAI for the infinite and the infinitesimal, and related number theory.

(5) On the construction of e

When the exponential function is introduced in COTP chapter 12, then there is loose reference to a function space. The number *e* arises as a fixed point in the map created by differentiation. This is a didactic approach that relies on the ease of such concepts. For a strict construction in this manner these concepts would have to be developed, including a proof for the fixed point theorem, and then there could be a vicious circle, a *petitio principii*, and an implicit reliance on limits anyway. However:

(1) The proper construction for *e* could use the more tedious route of the "formal approach" in paragraph 12.1.8.3, see COTP p173-174.
(2) While COTP develops the derivative in the algebraic manner, it is important to distinguish its didactics from the formal development.

See Colignatus (2017b).

The existence of a number *e* can be shown from $1^x < e^x <$ Lim[$b \to \infty$, b^x] rather than from $2^x > e^x < 3^x$ since the latter presumes more knowledge than is available yet.

(6) On the concept of function

COTP uses the concepts of function (domain, range, surjection, injection and bijection) but does not develop them. Domain and range are within the plane. COTP does not introduce set theory. The modern concept of function is no target of education with this book. Set theory is discussed in Colignatus (2011a) "A Logic of Exceptions" (ALOE), with the resolution of Russell's paradox, with indeed also consequences for Cantor, see FMNAI.

Cha (1999) (there are other authors on this topic, and they apparently tend to the same) reviews how the concept of function developed historically and how it can be applied in education.

(a) The original concept was algorithmic and concerned a formula how to calculate a result;
(b) then it shifted towards just dependence;
(c) and the modern concept is based in set theory (with emphasis on a univalent outcome).

For highschool education the latter appears rather abstract, and as a learning objective it is hardly attained because exercises concentrate on the algorithmic forms anyway.

COTP is fully in the latter two observations. However, COTP is not to be regarded as a particular argument about the position of functions within the total of the learning scope for highschool. COTP is only to be evaluated on what it intends to contribute about the news in COTP. Once the news is understood, we can compose a program that balances the various objectives.

COTP p72 has: "We say that y (the effect) is a function of x (the cause) when each x has precisely one value of y. We write $y = f[x]$."

COTP thus relies on the abstract modern approach (c) with a practical example of dependence (b). It is not wise to select only (a) or (b) or (c). It helps when "dependence" is clarified by a particular and important type of dependence.

Subsequently, COTP develops the derivative in algebraic manner. This means that COTP only looks at functions with a formula or algorithm, which is approach (a). Thus the learning scope of COTP seen on itself is restricted to such "algorithmic functions".

If set theory is introduced at another moment, the modern concept of function will likely be seen as a generalisation. Similarly the derivative for functions for which there is no such formula will also be regarded as a generalisation. This does not seem a drawback. If COTP succeeds in establishing the basic case then this seems a good result.

(7) Paradox versus inconsistency

A paradox is a "seeming inconsistency", i.e. something that seems like an inconsistency but that can be resolved by a closer look.

COTP holds: "The Weierstraß $\varepsilon > 0$ and $\delta > 0$ and its Cauchy shorthand for the derivative $\lim[\Delta x \to 0, \Delta f / \Delta x]$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken." Thus COTP does not hold that the Weierstraß and Cauchy methods are inconsistent. (No such inconsistency has been formulated yet.)

The Cauchy and Weierstraß methods however are paradoxical in this manner: *Conceptually* the derivative is taken at a point x but *analytically* that point x is only approached in the limit or surrounded by epsilon and delta. The algebraic way of COTP is to focus precisely on that point.

The paradox can be clarified with this diagram:

Common: $\Delta f = f[x + \Delta x] - f[x]$

Conceptual:

Derivative of a function at a point:

With function f[x] and derivative $g[x]$, the derivative of $f[x]$ at $x = a$ is given by $g[a]$.

Weierstraß:

General derivative df / dx:
For all $\varepsilon > 0$ there is a $\delta > 0$ so that
for $0 < | \Delta x | < \delta$ we have $|\Delta f / \Delta x - g[x]| < \varepsilon$.

Derivative at a point $x = a$: $\Delta f = f[a + \Delta x] - f[a]$
For all $\varepsilon > 0$ there is a $\delta > 0$ so that
for $0 < | \Delta x = x - a | < \delta$ we have $|\Delta f / \Delta x - g[a]| < \varepsilon$.

Contradiction
$\Delta x = 0$
(concept) and
$\Delta x \neq 0$ (method)

COTP:

$df / dx = \{ \Delta f // \Delta x,$ set $\Delta x = 0\}$

First the domain excludes 0, then it is extended with it, then it is reduced to it. Manipulate: R \ {0} then R then {0}.

This contradiction between concept and method is absent.

A computer program can't run through every epsilon to check the Weierstraß definition while the simplification of an algorithmic function might be done with algebraic methods (and likely the human mind also uses the latter method).

Equivalence between the two methods might only exist in that the same calculus rules are derived. The notions may also differ about what a "deduction" would be.

A special case involves a function for which the domain contains only a single point. A typical case is the classical conundrum: If average speed is defined as average distance divided by average time, then no such average time exists for the single moment in time that has a duration of zero. Parmenides would hold that speed is not defined for a single instance, whence it is not defined forever, whence motion does not exist. Let us now regard the Parmenides function $f[x] = x^2$ with the domain restricted to $x = 10$. Since domain and range are restricted to single values, Weierstraß cannot construct the epsilon and delta. Since Weierstraß focusses on numbers, the formula is of no use and we see only a ("constant") range of 100. The derivative is undefined or zero. With the algebraic approach we still can manipulate the formula and domain (e.g. conditionally or counterfactually), and find $2x$ and derivative 20 at that point. Clearly a function with a domain of a single point is pathological. We might say that for a single value the relation between formula and numerical outcome becomes awkward, though in fact this also would hold in general for all numbers (and if we collect this for all points it would hold also in the abstract). Whatever that be, we should say "derivative *of a function* at a point" and not just "derivative at a point". This pathological case clarifies why bisshop Berkeley had such problems with infinitesimals and why Cauchy felt a need to improve upon Newton and Leibniz, and Weierstraß felt a need to improve upon Cauchy. Berkeley did not have a case merely since the notions of number and function were still badly defined in those days: there really is a paradox (and conceptual inconsistency) on the division by zero when we use infinitesimals or limits.

One might hold that all definitions of the derivative at a point involve considerations of what the function does close to, but away from that point. In the algebraic approach the information about the area around the point is contained in the formula. In the Weierstraß approach the formula might play the same role, except that it is formulated in numeric form, and the modern definition of a function would also allow numbers instead of formulas. COTP rather sees it differently: The derivative that solves the classical conceptual issues is rather given by the algebraic method (COTP improving on Lagrange). The pathological case of a domain with a single point clarifies this. It hence becomes feasible to generalize from the algebraic approach to the Weierstraß format for cases that do not have an algorithmic formula. Didactics still require first the discussion of the algebraic approach before a case is considered for which that extension would be necessary.

PM. COTP essentially considers the didactic situation at highschool and the college or university freshmen years. It is not intended as a refoundational study in mathematics itself. Thus do not force it into a different straightjacket. The main point of COTP with respect to the derivative is that didactics cause and allow that it is not burdened with the limit and the infinitesimal. Nevertheless, in discussion on this, discussants bring up issues of non-continuity. Well, consider a function f such that $f[x] = x^2$ for $x \neq 2$ and $f[x] = 1$ at $x = 2$. The derivative at $x = 2$ is $\{(f[2 + \Delta x] - f[2]) // \Delta x$, set $\Delta x = 0\}$.

Then $\{(4 + 2.2.\Delta x + (\Delta x)^2) - 1) // \Delta x$, set $\Delta x = 0\}$,

or $\{(3 + 4\Delta x + (\Delta x)^2) // \Delta x$, set $\Delta x = 0\}$.

At $\Delta x = 0$ the dynamic quotient remains undefined.

My suggestion for a case like this is to return to the definition of what the derivative is intended for. The function f gives the surface under the derivative f'. Thus the surface under $f'[x] = 2x$ needs to be adapted in such manner at $x = 2$ that $f[2] = 1$ instead of 4. Thus a horizontal distance with width 0 is supposed to contribute a surface value of –3. A person who proposes such a 'mathematical problem' perhaps first ought to clarify what the meaning and intention of that 'problem' is, and whether it is not merely concocted to create a problem for its own sake. Still, one might consider $g[x] = x^2 + (f[x] - x^2)$ where the value –3 appears in the second part, define the general derivative on the uncontroversial part x^2 and for the pathological points include algorithmic statements such as: $f'[2] = c$ such that $f[2] = 1$. If it is a serious and non-concocted function then it might be better to have such an algorithmic statement than merely say that the derivative is undefined.

(8) The derivative of the trigonometric functions

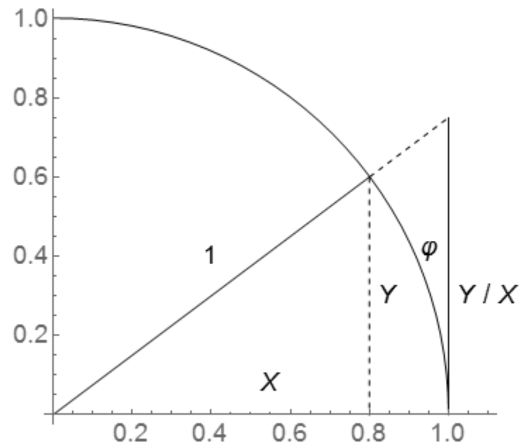
COTP p174 on the derivative on the trigonometric funcions better includes “The proof will be shown in the next paragraphs.” On page 175 the differential for Cos is worked out. There are two formula’s, the one for general φ and the one at φ = 0. The slope at zero is zero. This allows the deduction that (Cos[Δ*x*] - 1) // Δ*x* = 0 at Δ*x* = 0, so that part of the expression in the formula for general φ can be dropped. Then it is stated that Sin[Δ*x*] // Δ*x* = 1 at Δ*x* = 0, which is proven on page 177.

There is a distinction between the value for the function Sin and taking the derivative for Sin. The deduction on page 177 determines the value for Sin so that Sin[*x*] // *x* = 1 at *x* = 0. The method is a sqeeze in terms of logic. Thus $a \leq y \leq b$ and $a = b$ imply that $y = a$. There are no limits involved. COTP has the philosophy that a deduction is only done when needed. Perhaps this can result into the confusion that this derivative uses a squeeze and hence a limit. However, the deduction could be done separately and without a limit. Subsequently the derivative at *x* = 0 uses the value of the function for Sin[*x*] // *x* = 1 when *x* = 0.

Note 2017-01-25 & 2017-02-13:

The deduction in COTP on page 177 may be needlessly complicated by involving the areas, though the areas actually might help to get a clear structure with $a \leq b \leq c$.

It can be done quicker as follows. There is no need to refer to Sin and Cos - as COTP also standardly uses {*X, Y*} on the unit circle. Regard the unit circle with *Y* = Sin[φ] and *X* = Cos[φ]



We find the length *Y* / *X* because [1] the ratio *Y* : *X* is proportional to (*Y* / *X*) : 1. Thus:

$$Y \leq \varphi \leq Y / X$$

The dynamic quotient *Y* // *Y* = 1 on the left and (*Y* / *X*) // *Y* = 1 / *X* on the right are without problem.

Then the dynamic quotient generates 1 ≤ (φ // *Y*) ≤ 1 / *X*.

And from this: 1 ≤ {φ // Sin[φ], set φ = 0} ≤ 1.

[1] http://aleph0.clarku.edu/~djoyce/java/elements/bookVI/propVI13.html and https://www.quora.com/Is-there-a-purely-geometric-way-of-doing-division

Conventional differential analysis has $1 < (\varphi \,//\, Y) < 1 / X$, and thus inequality rather than the possibility of equality. The deliberate use of inequality forces the need for limits.

Including the use of equality allows to squeeze in *logical* fashion. If $a \leq b \leq a$ then $b = a$.

Note 2017-02-12:

Streamlined:

Step 1: $1 \leq a \leq 1$ causes the logical conclusion that $a = 1$.

Step 2: Investigate a = {φ // Sin[φ], set φ = 0}.

The RHS can not be analysed further. But we may try to find a numerical value for it.

Step 3: $Y \leq \varphi \leq Y / X$ from the diagram above

Step 4: The dynamic quotient $Y \,//\, Y = 1$ on the left
and $(Y / X) \,//\, Y = 1 / X$ on the right are without problem. Thus:

$1 \leq (\varphi \,//\, Y) \leq 1 / X$

Step 5: 1 ≤ {φ // sin[φ], set φ = *k*} ≤ {1 / cos[φ], set φ = *k*} (as a function of *k*)

Step 6: k = 0

Step 7: 1 ≤ {φ // Sin[φ], set φ = 0} ≤ 1

Result: {φ // Sin[φ], set φ = 0} = 1

Note 2014-04-19:

Just to be sure: p177 contains the step "Dynamic quotient by Sin[φ]...". This might give the impression that all sides are divided by Sin[φ]. (PM. We only consider nonnegative values of φ, conforming to the graph (otherwise the inequalities would flip).) But it is not intended that there is mere division, since one cannot divide by zero. The point is that we have an interest in the dynamic quotient. Thus, as we have $a \leq y \leq b$ and wonder what the dynamic quotient i.e. $y \,//\, x$ is, then we can use that *there is already a division*:
Rule: For $y \,//\, x$ and $x \geq 0$: if $a \leq y$ then $(a \,//\, x) \leq (y \,//\, x)$.
On p177 the particular case is $a = x$ = Sin[φ]. And separate application to b.

PM. If you would want to construct a counterexample such that the inequality sign flips when $x = 0$, by choosing a or y with a switch at *zero,* then you fall into the trap that our x no longer is a variable at that switch value but actually a constant, and you cannot divide by zero.
An example of such an inappropriate counterexample would be:
For $x \geq 0$:
$a = 10\, x$, with $a \,//\, x = 10$.
y = {if $x = 0$ then x else 100 x}. Then $y \,//\, x$ = {if $x = 0$ then 1 else 100}.
We have $a \leq y$ (both zero at $x = 0$).
Now at $x = 0$: $(a \,//\, x) = 10 > 1 = (y \,//\, x)$.
The error in this "counterexample" is that you should write
not y = {if $x = 0$ then x else 100 x},
but y = {if $x = 0$ then 0 else 100 x}
so that the $y \,//\, x$ gives 0 / 0 at $x = 0$, since there is no proper variable.

(9) 2012-06-21: Addendum on Abs[x]

COTP:160 states that |*x*|' = sgn[*x*]. In retrospect it is better (a) to include a statement here that this is rather a matter of definition and elegance, (b) to include a longer discussion in the back like the following.

COTP:160 uses |*x*| = sgn[*x*] * *x.*

The deduction for the derivative at 0 is:

df/dx = {(|0 + Δ*x*| - |0|) // Δ*x* = |Δ*x*| // Δ*x* = sgn[Δ*x*] * Δ*x* // Δ*x* = sgn[Δ*x*], then Δ*x* = 0} = 0

where the dynamic division first assumes R \ {0} and after simplification extends the domain with {0}.

It however is also possible to take *x* = sgn[*x*] * |*x*|.

Then |Δ*x*| // Δ*x* = |Δ*x*| // (sgn[Δ*x*] * |Δ*x*|) = 1 / sgn[Δ*x*], and then Δ*x* = 0 gives indeterminacy.

Commonly if there is one way to determine indeterminacy then this takes precedence.

Thus |*x*|' = 1 / sgn[*x*] instead of |*x*|' = sgn[*x*].

> This can also be shown in this manner:
>
> Define the function sr[*x*] = sgn[*x*] + *r*[*x*] where *r*[0] is a random number and *r*[*x*] is zero elsewhere.
>
> Then |*x*| = sr[*x*] * *x.*
>
> The deduction for the derivative at 0 is:
>
> ***df/dx*** = {(|0 + Δ*x*| - |0|) // Δ*x* = |Δ*x*| // Δ*x* = sr[Δ*x*] * Δ*x* // Δ*x* = sr[Δ*x*], then Δ*x* = 0} = *r*[0]
>
> Thus the derivative is indeterminate at 0.

Now, however, my suggestion again is to return to the definition of what the derivative is intended for. The function *f* gives the surface under the derivative *f'*.

For sr[*x*] the value at 0 has no impact on the surface. There is no harm done and it is quite elegant to resolve the indeterminacy by taking |*x*|' = sgn[*x*]. This then becomes a matter of explicit definition.

Admittedly, though, this approach to see matters from the viewpoint of surfaces then differs deliberately from the approach of considering slopes, and this will require more discussion than the simplicity of COTP:160 now suggests.

[see below]

(10) 2013-12-01: Addendum

(1) Slopes again

COTP:68 defines slope, and has a long discussion on it. COTP:159 discusses whether the derivative can be interpreted as the slope. Traditionally slope and derivative are identified as the same. This is proper when the emphasis is put upon slope. However, when the emphasis is put upon surface, we may reconsider this. The example is Abs[x].

COTP:160 and these Reading Notes above discuss for Abs[x] that we can hold that the surface under sgn[x] does not change at x = 0, so that we may take Abs'[x] = sgn[x] in general. The latter would require an independent definition of slope and derivative.

The independence arises naturally by saying that: inclination gives the line called "incline".
(a) The slope is the "coefficient of inclination".
(b) The intercept is the other parameter of "inclination".
Given this independent definition of "inclination", it causes the natural question for a general function f[x] what would give the inclination at a point {a, f[a]}.

COTP:159 can be enhanced with the line f*[x] through {a, f[a]}:

{(f[x] - f[a]) // (x - a), set x = a} = s = f'[a] gives f*[x] = f[a] + s (x - a)

When h = x – a can be dynamically divided out of the quotient, then
(a) we can say that "the function at a and the line have the same inclination"
(b) we can say that "the line is a tangent to the function", with the explanation:
(c) we can make a graph such that s = yur[alpha] / xur[alpha] = tur[alpha] i.e. the tangent.

These ideas are already received. COTP focusses on presenting the new ideas and not on repeating received ideas. However, this would seem to be the best place to put these ideas.

Now, with independence between slope and derivative, we have:
(i) looking for the inclination, we must have Abs'[x] = 1 / Sgn[x]
(ii) looking from the surface, we may choose Abs'[x] = Sgn[x] if this would be advantages for some applications.

PM. COTP:149 in clarifying calculus mentions "change in surface" right at the start. This is too fast and needlessly confusing. This starts with surface and it suffices to say so. After three pages the change in surface is discussed, and then it can be said tha calculus also deals with that.

(2) Lagrange

Norman Wildberger on Lagrange's algebraic method for the derivative of analytic functions:

http://www.youtube.com/watch?v=oW4jM0smS_E

2014-09-11: I read Judith Grabiner (2010): "A historian looks back. The calculus as algebra and Selected Writings", MAA. She gives lots of attention to Lagrange. I had wanted to say more on this but ..... For now:

http://boycottholland.wordpress.com/2014/06/29/euclids-fifth-postulate
http://boycottholland.wordpress.com/2014/06/14/amir-alexander-and-history-as-storytelling

(3) *g*[*x*, Δ*x*]

The difference quotient is actually a twovalued function: $g[x, \Delta x] = (f[x + \Delta x] - f[x]) / \Delta x$.
This function is undefined for $\Delta x = 0$.
By regarding *x* as a parameter, it is reduced to a single-valued function again.
If $f'[x]$ =Lim[$\Delta x \to 0$, $g[x, \Delta x]$] exists then it is called the derivative at *x*.
COTP explains that this involves a manipulation of the domain, and that limits are not necessary when we allow for algebraic simplification. (See also the video now.)
In highschool, we are not allowed to speak about twovalued functions however.
When we start with surfaces then this is less of a problem.

However, also there, and actually overall, it will be useful to mention the possibility of multivalued functions.

PM 2017-01-23. Joost Hulshof pointed to his approach to integrals, but my impression is that COTP is easier to understand at highschool: http://www.math.vu.nl/~jhulshof/reader2.pdf
Hulshof actually doesn't show the algebraic approach, and in a bilateral email exchange it appears that he might calculate the derivative but doesn't show that it is a slope. See the longer discussion about Range (2016) "What is Calculus?"
Hulshof also mentioned that Apostol starts with surface. I haven’t checked this: http://en.wikipedia.org/wiki/Tom_M._Apostol

(4) On angles, xur and yur

(a) Lecture by Joost Hulshof

In this lecture, Joost Hulshof uses X and Y for cos and sin. These are only intermediate variables, so that he does not make the conceptual jump to use {X, Y} as new co-ordinates to introduce trigonometry. However, it is nice to see that he is close to doing so. My impression is that there may be many examples by other authors. It is rather natural to select the variable names in this manner. I did so too, but then wondered about the didactic advantage, and came upon xur and yur.

http://www.few.vu.nl/~jhulshof/nawdec270.pdf

page 273, or at the journal website itself:

http://www.nieuwarchief.nl/serie5/toonnummer.php?deel=08&nummer=4&taal=0

http://www.nieuwarchief.nl/serie5/pdf/naw5-2007-08-4-270.pdf

PM. 2016-11-30: On trigonometry, see: http://thomascool.eu/Papers/Math/2016-09-04-Trig-Rerigged-2.pdf

(b) Norman Wildberger YouTube MF38: Why angles don't really work (I, II and III)

I agree with Wildberger about the complexity of the standard introduction to trigonometry but I have to think a while about his suggestion of using “spread”.

http://www.youtube.com/watch?v=j7bxL2HgZbk&list=PLA6E556990EE0C52A&index=38

http://www.youtube.com/watch?v=4-h-TPmRWZc&list=PLA6E556990EE0C52A

http://www.youtube.com/watch?v=YvHaaQv-kfg&list=PLA6E556990EE0C52A

His alternative is: spread[angle] = sin[*degree*]^2 = yur[*angle*]^2, see http://www.youtube.com/watch?v=9wd0i44vK04&list=PLD6BAC8FA4287A3DD

The advantage of “spread” is that working with squares avoids surplus calculations in intermediate steps (taking square roots and squaring again). In some standard cases there are no square root signs.

The disadvantage is that you lose the conceptual clarity that xur and yur are co-ordinates, and that angles are turns (or “unit measure around”). Taking the square root is a natural step from going from surface to length. Thus it cannot be objected to conceptually. It is only awkward if it can be avoided in calculation indeed.

Didactic experiments ought to show what works best for students.

Potentially for measurement issues, it might be a solution though to have rational values only.

(11) 2014-09-11: (a) Sky over field & meadow, (b) Pierre van Hiele & David Tall

There are now two articles.

(1) COTP is built upon the Van Hiele theory of levels of abstraction. It also uses the procept by Gray & Tall. The following paper enlightens the situation w.r.t. these theories and theorists.

Colignatus (2014a), "Pierre van Hiele and David Tall: Getting the facts right", July 27, http://thomascool.eu/Papers/Math/2014-07-27-VanHieleTallGettingTheFactsRight.pdf

(2) Following a discussion with Jan Bergstra: (a) there may be more formal development for the dynamic quotient in terms of group theory, as a sky over a field or a meadow, (b) in the definition of the dynamic quotient the replacement of "a variable" by "(a) variable" might be more robust against confusion about substituting expressions. Using the Harremoës operator H = -1 and D for the label of dynamic division:

$y\, x^D \equiv \{\, y\, x^H$, unless x is (a) variable and then: assume $x \neq 0$, simplify expression $y\, x^H$, declare the result valid also for the domain extension $x = 0$ }.

Coliignatus (2014b): "Education, division & derivative: Putting a Sky above a Field or a Meadow. Comments on the field, the meadow, the dynamic quotient and the derivative, as seen from education in mathematics (elementary, highschool & matricola)", http://thomascool.eu/Papers/Math/2014-09-08-Sky-Field-Meadow.pdf

Note that one must write brackets otherwise the variables might wander: $(y\, x^D)$

(12) 2016-11-30: Continuity, state of affairs, FMNAI

- For continuity, see Colignatus (2016c). This is actually a major step.
- For an overview of the state of affairs in 2016, see Colignatus (2016ab).
- An important new text is "*Foundations of Mathematics. A Neoclassical Approach to Infinity*" (FMNAI) (2015). Overlap with the derivative is: we still don't need infinitesimals.

(13) 2016-12-11: Finally a sound interpretation for differentials

Up to now, differentials were superfluous.

In COTP and in the above, $df\,/\,dx$ was only used to link up to the literature, but there wasn't a separate development, yet.

Differentials are useful, in particular for partial derivatives. Up to now I didn't have time or inspiration to collect some thoughts about the traditional treatment of differentials. Now I did, see Colignatus (2016d). The discussion appears to be a major step and welcome supplement. It should become much easier for readers to understand the algebraic approach.

There is a clear distinction now between the mess in the past and the clarity provided now.

Historically, differentials were seen are part of the world of infinitesimals, alongside the epsilon and delta. However, differentials can take any value. The use of differentials might perhaps be called "analysis of variation".

The general format is "true value = estimate + error". Now we have $\Delta f = s\, \Delta x + \varepsilon$ where $s \equiv \{\, \Delta f \,//\, \Delta x$, set $\Delta x = 0\}$. The common attention is for $\Delta f \,//\, \Delta x$ but Weierstrasz seems to have switched attention to the error, remaining in the world of the infinitesimal. Perhaps it is useful to develop this line of argument.

(14) 2017-02-17 Comparing with WIC Prelude, Rational Functions and "direct approach"

Colignatus (2017e) compares with a paper by Shen & Lin (2014), that also develops integral and derivative in simultaneous manner.

Colignatus (2017d) looks at the theory of "rational functions".

Colignatus (2017e) compares with Michael Range's "What is Calculus?" (WIC) and in particular the WIC "Prelude".

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