REACTION TO HART (2015) ABOUT CANTOR'S DIAGONAL ARGUMENT

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Hart (2015) reviews Cantor's "Diagonal argument". He presents a view that after the support by Hilbert may be called "traditional". Readers may benefit from Van den Berg (2013) for a perspective on Hilbert. It so happens that a refutation can be found in my book *"A Logic of Exceptions"* (ALOE) (1981, 2007, 2011) which is neglected by Hart. See a review by Gill (2008) in this journal. Later I updated specifically on Cantor's diagonal argument in *"Contra Cantor Pro Occam - Proper Constructionism with Abstraction"* (CCPO-PCWA) (2012, 2013) on which I informed Hart. In November 2014 - June 2015 I gave an update on the relation to the ZFC axioms. This has become PV-RP-CDA-ZFC (2014, 2015) and *"Two results on ZFC"* (2015e). Let me first state my comments w.r.t. Hart (2015) and then reflect on the discussion that these comments caused.

(1) Hart (2015:43) holds correctly that a bijection doesn't have to be used, but only the surjection (i.e. in the mode of thought that the proof would be valid). He however holds incorrectly that the common short proof with the bijection would rely on a 'spurious contradiction' - referring here to Gillman 1987. This would be incorrect if we rely on the common meaning of 'spurious': (a) there is a real contradiction: the assumption of the bijection implies the assumption of the surjection, which causes the contradiction, (b) the context of discussion is infinity, for which we use isomorphisms, and thus injections, and in that case the properties of surjection and bijection are equivalent: and then the shortness of the proof must be appreciated. Indeed Hart (2015:41) explains that Cantor himself also used 'eindeutig' (column 1) and injection (column 3 - below the photograph of 'Georde Cantor'). Overall: the open 'reductio ad absurdum' form and the 'direct' form that Hart suggests are equivalent, and the reference to 'spurious contradiction' is incorrect. PM. Hart (2015:42 first column) suggests that the power set version of Cantor's Theorem was given by Bertrand Russell 1907, also using a 'supposition' and basically using a bijection.

(2) On page 42, third column, Hart agrees that Cantor's distinction between proper sets and improper sets ('classes'), or the distinction between *all* and *any*, still is used informally. Thus mathematics uses both a formal ZFC and an informal naive set system. It is useful to see this confirmed. It remains curious that Hart as a mathematician is happy to live with this incongruity. Hart then discusses the axiom of separation, but it gives a wrong impression, because its main weaknesses and alternatives are not discussed. One may write a book or syllabus on 'set theory' but if this only discusses ZFC and its ZFC-sets then this is a biased presentation.

(3) On page 43 Hart mentions the argument concerning $\aleph \sim \Re$ that uses decimal expressions. He states that this particular form does not occur in Cantor's work. This is not quite true. Cantor's proof of 1890/91 uses a binary representation - see Hart (2015:41) -

which, for these purposes, is equivalent to using decimals. Hart traces the proof with decimals to Young & Young in 1906, who explicitly refer to Cantor 1890/91, and who explicitly call it his 'second proof'. Thus mathematicians were aware already in 1906 that binaries and decimals are equivalent here. It is curious that Hart in 2015 does not express that awareness. His review of what Cantor originally did thus is biased. (3a) For this proof structure, binaries and decimals are equivalent. (3b) The binaries are mathematically more elegant, since changing an element has only one alternative. The decimals are didactically more useful, since students are more used to decimal expression of the real numbers - which is the representation of the continuum. It would be improper to criticize the decimal form of the proof for being didactic. (3c) It is correct that Cantor claimed that the proof structure was "independent from looking at irrationals" but the proof does *implicitly use* irrationals.

(4) We may wonder why Hart's paper might be biased. It is a good hypothesis that he wants to emphasize that some authors still have questions about Cantor's argument.

(4a) On page 43 Hart refers to Wilfrid Hodges (1998) who discusses "hopeless papers". Hart does not mention Hodges's email to me that I cited in CCPO-PCWA that I informed him about.

(4b) Hart accuses those "hopeless papers" of that they don't check what Cantor did himself originally. This is an improper accusation since such authors discuss a particular argument, that so happens to go by the name of "Cantor's diagonal argument", while it is not always at issue what Cantor himself did - who indeed wrote before ZFC.

(4c) Just to be sure: My own first contact with Hart - in 2011 - was about Cantor 1874. CCPO-PCWA wanted to know whether there were more proofs, and thus also looked at Cantor 1874, and found it inadequate. Hart's page 40 with Cantor 1874 finds a refutation in the appendix of CCPO-PCWA - but he knows about the latter and does not refer to that refutation.

(4d) Hart suggests that the proof with decimals causes most "hopeless papers", but that this proof can be "thrown in the trash can", because Cantor's original proof from 1874 and his second and more general proof of 1890/91 would be more attractive.

(4d1) This is improper, since it evades the question whether the argument with the decimals is a good deduction or not. Mathematics should not ditch arguments because they cause questions but should answer the questions.

(4d2) It also is an inconsistent argument, see (3): the proofs are equivalent, differ only in binaries versus decimals. Thus Hart suggests to throw Cantor's own proof into the trash can - but doesn't do so.

(4d3) In a personal communication, Hart acknowledges that the binary and decimal proofs are equivalent (without drawing the inference on (3)) but that he only expressed his preference for the aesthetics of the binary form. He is free to state his preference, but the decimal form is the most didactic one, and thus the form *cannot* be ditched.

(4e) Hart holds that such "hopeless papers" and/or internet discussions quickly replace mathematics by ad hominem fallacies. An ad hominem would be: "You have no mathematics degree and hence I will not listen to your arguments." Obviously Hart presents himself as not falling into that trap. My problem however is that he applies an 'ad gentem fallacy', by reducing critique on Cantor's Theorem into "hopeless papers" and/or internet ad hominem fallacies. This is a racket or ballyhoo to induce a sentiment amongst his readership to no longer look at critique on Cantor's Theorem, and to join in the putting down of such critics. We thus may understand why Hart (2015) is a biased presentation, unworthy of mathematics that wants to claim to be scientific.

(5) Hart (2015:42, last column): "The best known impossibility theorems in mathematical logic all use a version of Cantor's idea to flip all elements on a diagonal" - and then he refers to Gödel's first incompleteness theorem. This is not quite true. Gödel's theorem uses self-reference. This property was already known in antiquity in the Liar Paradox. Gödel's use of number-coding has historical explanations, like the trust in arithmetic in a period of a foundations crisis in mathematical logic. Gödel's numerical listing is not crucial to the argument. The influence of Cantor should not be made greater than it is. Hart could have known about this, reading both ALOE and Gill (2008) in the same Dutch journal for mathematics, with my refutation of Gödels two theorems.

(6) Hart does not refer to ALOE or CCPO-PCWA that he knows about, thus misinforms his readership. He reproduces Cantor's 'proofs' of 1874 and 1890/91 without mentioning their refutations. He states the common misconceptions and adds some new ones.

The above were comments on Hart (2015). In a personal communication, Hart has looked at my criticism. It leads too far to look into this here. Colignatus (2015a) reviews the email exchange with K.P. Hart (TU Delft) in 2011 - May 2015. Colignatus (2015b) reviews Hart's response on the above. The reader can check that the criticism still stands. Colignatus (2015c) reviews Hart's new combined criticism of May 18 2015 on that version of PV-RP-CDA-ZFC, and earlier refutations in CCPO-PCWA - which should cover point (6). There is now also the issue on scientific integrity, see Colignatus (2015d).

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