Abstract of *key new results* in "A logic of exceptions" (ALOE)

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"A logic of exception" is a 240 page book and course in Elementary Logic, all available within the interactive environment of *Mathematica*. Proofs and exercises are provided for by program calls that the user can apply at will. The PDF and the software are available for inspection at http://www.dataweb.nl/~cool/Papers/ALOE/Index.html.

The book presents:

(1) the basic elements: propositional operators, predicates and sets;
(2) the basic notions: inference, syllogism, axiomatics, proof theory;
(3) the basic extra's: history, relation to the scientific method, the paradoxes.
(4) The new elements in the book are: a logic of exceptions, solutions for those paradoxes, analysis of common errors in the literature, routines in *Mathematica*.

The book is intended to be used in the first year of college or university. The last two chapters require a more advanced level that is worked up to.

This abstract concentrates on the ***key new results.*** These are to be distinguished from the accessible reformulations of known results and from the implementations in *Mathematica*. This abstract is intended for a teacher in logic or an advanced student of logic, who considers himself or herself advanced above Elementary Logic, and who wants to have a clear grasp of the key news in the book. This teacher or advanced student might concentrate on the 50 pages mentioned below to focus on the key new results.

The ***key new results*** for logic and the foundations of mathematics are:

> The Zermelo-Fraenkel axioms for set theory can be revised and Gödel's theorems apparently don't hold (or only partly for some limited assumptions). The basic logic that we should be using is three-valued and not two-valued. The "levels in language" as seen by Russell and Tarski then aren't required, and nonsense expressions like the Liar paradox and the Gödeliar disappear in a "sink" of the third logical truth value, not True or False but Indeterminate. Finally, the basic scheme in all inference is that one always has to keep in mind that there may be exceptions.

The cover of ALOE states: "Confronted with the Liar paradox ("This sentence is false"), modern logic recognizes three main approaches: **(1)** the theory of types, with levels in language, **(2)** the abandonment of truth and adoption of a theory of proof with undecidability instead of inconsistency, and **(3)** three-valued logic. The first two approaches sacrifice useful forms of selfreference. Also, the Gödel incompleteness theorems are based upon a cousin of the Liar ("This sentence is unprovable"), which makes the result rather unconvincing, while the philosophical implications on undecidability seem too large in relation to the dubious nature of such a paradoxical statement. Hence three-valued logic is a more fruitful approach. The logic community has hesitations about three-valued logic. The programs implemented in *Mathematica* show that it works quite nicely."

Let us consider the various elements.

(1) The Liar paradox is discussed on pages 114-116 and then again in the chapter 7 on three-valued logic on pages 191-193. A key new result is:

> Let L = Not(L). Given the induced contradiction in two-valued logic we conclude that for all x in two-valued logic it would hold that $x \neq$ Not(x). Russell and Tarski solved that by a theory of types in language. Yet this causes truth-predicates true_n for language level n which is rather odd since natural languages don't seem to have use for this. If we allow for three-valued logic then we keep the freedom of definition and a single concept of truth.

(2) Three-valued logic gets a chapter on pages 177-194. Key new results are:

(a) In two-valued logic we have the "convention of the assertoric usage of language", in that we say only what we consider true. With two values of saying (saying and not saying) and two values of truth (true and false) this fits. What happens when there is a third value ? The solution is to recognize "hypothetical speech" as a third possibility in the use of language, e.g. used in a "reductio ad absurdum".

(b) The paradoxes of threevalued logic appear to be solvable by repeated application of the third value operator. The crux appears not the addition of new truth values but the inclusion of new rows in the truth tables.

(c) While three-valued logic seems cumbersome at first, the routines in *Mathematica* show that it can be handled easily. And it is only relevant in cases when one deals with nonsense.

(3) The discussion of Russell's set paradox can be found on pages 95, 121 and 127-129. A key new result is:

> Russell's set paradox comes with $R = \{y \mid \neg(y \in y)\}$ where "ε" stands for being an element and "¬" for "not". Given the induced contradiction in two-valued logic we conclude that for all sets x it would hold that $x \neq \{y \mid \neg(y \in y)\}$. Russell and Whitehead solved that by a theory of types, so that it is ingrained in the Zermelo-Fraenkel axioms of set theory and in the syntax that such an R cannot be formed. The syntax allows for instant deduction, and forbids selfreference. Let us however consider the close approximation of the R-idea in $Z = \{y \mid (\neg(y \in y)) \ \& \ (y \in Z)\}$. In this case we allow selfreference and break the theory of types. This set is inspired by Zermelo's axiom of *Aussonderung*, and identifies that selfreference is the working part of that axiom. Thus, it is useful to allow for selfreference and to rewrite the axioms. The problem is solved by inference and not by syntax, allowing for more freedom of syntax. Since this allows the formation of R, that expression can be regarded as a nonsensical concept, such that the expressions $R \in R$ and $\neg(R \in R)$ get truthvalues Indeterminate instead of True or False. In this case we rely on inference rather than syntax to prevent paradox, with the premium that we can define sets using selfreference when it is useful. Note that Z can be

understood as $Z = \{y \mid (\neg(y \varepsilon y))$ ***unless*** $y = Z$, ***for then*** $(\neg(y \varepsilon y)) \;\&\; (y \varepsilon Z)\ \}$ which means that R is equipped with an exception.

ALOE does not provide new axioms for set theory, it only points to a solution approach. For set-theorists it might require some getting-used-to to not have an axiom here but merely the logical deduction that expressions like Russell's can be found to be nonsensical.

(4) Gödel's theorems are discussed in chapter 9 on pages 205-226. A key new result is:

> In this case, wild philosophical conclusions have been based upon a sentence that is a cousin of the liar sentence, the Gödeliar "This sentence is not provable". These wild philosophical conclusions still might be true, but not based upon this nonsensical Gödeliar. First it is strange that logicians recognize the relationship between truth and provability, and that they require levels for truth, but drop this for provability. Gödel's numerical coding allows selfreference, as this already exists in language. However, he only allows selfreference for sentences and not for systems. The solution is to allow selfreference for systems too. Consider the notion of "proof-consequentness" that "when something is proven, then it is also proven that it is proven". Or for a person: "When I say something, then I will also say that I said it." This property is a bit complex since the Gödeliar is similarly complex. When we equip the axiomatic system with that rule, we can derive the same contradiction as with the Liar paradox. Gödel's theorems are based upon nonsense, and we need only weak additional properties to show this.

The property of proof-consequentness is not by itself included in the original axioms for (Peano-) arithmetic but, it can be added, and it makes sense to add it when considering issues of proof-theory. The meta-mathematical suggestions by Gödel's theorems then evaporate.

Taking (1) to (4) together: for the first time since Eubulides of Milete (ca. 350 B.C), a pupil of Euclid, presented the Liar paradox, there is full clarity about its solution.

NB. ALOE was written 25 years ago in 1980-1981 when the author was still a student of econometrics. At that time the professor in logic did not like the analysis, didn't want to give a grade on it, and since this student in econometrics needed to graduate in econometrics he dropped the subject and shelved the book. The author has now retyped his original manuscript, put it in the *Mathematica* environment and extended it with the logical routines.

There is one paragraph in the book that can usefully be copied here, stating another key new result, that however appears to have found its way into the literature via other means:

> "In February 2007, correcting the typing errors in the January 2007 print test run, I can add the following. I contacted the professor of 1980-1981, informed him of the PDF and software on the web, suggested that the software might make a difference so that he might better understand the analysis, added that my check of the analysis confirmed it again, invited him to try again, and offered to give him a password for that software to do so. He didn't react. It also appears that he has been publishing a lot on dynamic logic and getting huge research grants for it. In 1980-1981 this was the only thing that he said that he liked in my papers - and when we discussed it back then it was also obvious that he hadn't thought about that angle himself. I don't know whether he gives proper reference now. It is useful for me to add this comment otherwise people might think that I don't give proper reference to him. For the record: the distinction between statics and dynamics comes from economics and in 1980 it felt like a natural interpretation for me to apply it

> for the distinction between propositions and inference. It is only a small element in the whole of this analysis."

PM. The author currently studies the links between logic and causality. His hypothesis is that causality cannot be inferred from common statistics and has to do with the model and the order of calculation (time's arrow). In logic, there is a difference between (static) implication => and (dynamic) inference $\vdash$. In causal models, there is a difference between (static) equality = and (dynamic) assignment :=. What causality is in Nature, inference is in the Mind. The overall umbrella would be the difference between statics and dynamics. This needs to be elaborated on however. If ALOE hadn't been blocked 25 years ago, this could already have been done.

Literature

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