Contra Cantor Pro Occam

A discussion of my books ALOE, EWS and COTP with respect to Cantor's Theorem and non-standard analysis, i.e. the issues of infinity and infinitesimal

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NB. Henceforth it will be more convenient to refer to this present paper as CCPO - WIP (work in progress). Namely, this present paper has become rather lengthy and it is useful to split off subpapers.

There is now "Contra Cantor Pro Occam - Proper constructivism with abstraction" (March 2012) (CCPO-PCWA) that is CCPO rewritten without infinitesimals and the exposition about the genesis of the analysis in CCPO.

See http://www.dataweb.nl/~cool/Papers/ALOE/2012-03-26-CCPO-PCWA.pdf

Summary

A consequence of "A Logic of Exceptions" (ALOE, draft 1981, 2007, 2nd edition 2011) is that it refutes 'the' general proof of Cantor's Theorem on the power set, so that the latter holds only for finite sets but not for "any" set. The diagonal argument on the real numbers can be rejected as well. There is a bijection 'in the limit' between ℕ and ℝ, which can also be called a bijection 'by abstraction'. If no contradiction turns up it would become feasible to use the notion of a 'set of all sets' 𝕊, as it would no longer be considered a contradiction that the power set of 𝕊 would be an element and subset of 𝕊 itself. The books ALOE, "Elegance with Substance" (EWS, 2009) and "Conquest of the Plane" (COTP, 2011) also develop calculus without the use of limits or infinitesimals. Lagrange's algebraic approach is best supplemented with a manipulation of the domain. Non-standard analysis is not needed for the derivative. Some aspects of it may be reformulated and then may be of use for the education on the real number line. See "Neoclassical mathematics for the schools" (2011). This paper puts these results into historical perspective. The Appendix discusses Cantor's original 1874 proof and suggests that the notion of a limit in ℝ cannot be defined independently from the construction of ℝ itself.

PM. I thank Richard Gill, Bart van Donselaar, Alain Schremmer and J. John C. Kuiper for some comments and suggestions for relevant literature. I thank two other readers

whose comments have resulted in the following two corrections. In February and March 2012 a confusing typing error in section 2 (ϕ instead of φ) was corrected. In March 2012 the exception for Russell's and Cantor's sets that was already explained in ALOE (2007:129) was stated here again, since the shorthand notation created confusion on infinite regress.

1. Introduction

1.1 Rigour, with footnotes

Set theory belongs to logic because of the notions of *all, some* and *none*, and it belongs to mathematics once we start counting and measuring. Cantor's Theorem on the power set somewhat blurs that distinction since the general proof uses logical methods while it would also apply to infinity - and the latter notion applies to the set of natural numbers $\mathbb{N}$ = {0, 1, 2, ... } and the set of reals $\mathbb{R} = 2^{\mathbb{N}}$. It is a mathematical abstraction that infinity comes about when a set cannot have a bijection with all its subsets, but to show the existence of such a possibility we tend to refer to the natural numbers.

Since a mathematical paper requires a theorem and proof, this paper will prove that Cantor's Theorem does not hold for infinite sets. This approach uses three-valued logic, as explained in my "A Logic of Exceptions" (ALOE, draft 1981, 2007, 2nd edition 2011). See Gill (2008) for a review of ALOE and Gill (2012) for a review of EWS and COTP.

Logic and number are also difficult to separate when we link up with geometry. Consider a line tangent to a circle. They have a point in common. A point has no size but only position. Thus line and circle have 'nothing' in common, in terms of size. They have 'something' in common, in terms of position. But point/nothing ⇔ position/something. To have nothing and something in common would be internally inconsistent. The solution is to regard the issue in terms of logic and different perspectives. The solution requires abstraction. In one respect (size) it is nothing and in another respect (location) it is something. A size measure without paradox can then be found in the distance from the origin. The solution requires a mix of logic, number and geometry, and we meet the same mix in issues of the infinite and infinitesimal. Only careful distinction of perspectives will keep us out of contradictions.

My earlier work tended to regard ∞ and $1 / \infty$ as undefined, except only as shorthand. The paradoxes of infinity thus considered essentially solved, I tended to neglect those. (A paradox is only a seeming contradiction.) Somehow and rather unwillingly there arose some trail of thought that forced me to look into these paradoxes. This paper reports about this foray into number theory and non-standard analysis, starting from the sound basis in DeLong (1971). The results are tentative and I am pretty sure that there are experts who have much more to say on this. As an econometrician and teacher of mathematics without a background in number theory and non-standard analysis I have only accessed some introductory works and history books that I thought that I could understand. My presumption was that the easy books could transfer the fundamentals in a clear manner. They didn't. My presumption was that it would be a waste of time to look into the hard stuff if the introduction was already unclear. This is not generally true of course. In economics it might take various deep courses to be able to read the newspaper, i.e. to see why most newspaper reports are bogus. But numbers ? For highschool ?

Boyer (1949) on the history of calculus provides 'rigour, with footnotes'. That is, his chapter VII "The rigorous formulation" includes footnotes (specifically no 77 and 92) at crucial points - "Now the fundamental theorems on limits could be proved rigorously [77]" and "(...) which clarifies the situation [92]" - which footnotes destroy the suggested rigour and clarification.

Given the tentative nature of this paper, otherwise than the proof on Cantor's Theorem, I record questions and suggested answers. For example, when introductory books and papers on number theory and non-standard analysis speak about a single notion of limit (though defined in various manners, e.g. a limit to a terminus or really attained), then I not only record that confusion, but my suggestion is also that there are (at least) two notions, one for decimal *number* and one for measured *length*. It is not clear to me yet whether this distinction makes full sense but at least it helps to understand different perspectives in a literature that is otherwise rather confusing.

The infinitesimal enters mathematics via Zeno's paradoxes and our solution via calculus. The history of calculus apparently has these phases: (a) Zeno's paradoxes and the ancient solution of exhaustion, where Aristotle helped Euclid to evict Democritus's atom from the notion of space itself, such that points have no size and lines have no width, (b) Newton and Leibniz with infinitesimals, (b) Euler and Lagrange with algebra, (c) Weierstraß with ϵ and δ. Boyer (1949) clarifies that Leibniz used all methods in somewhat confusing ways so that he cannot be claimed for one particular direction. Taking this history into account, my reading of the current literature anno 2011 is:

(a) Logic leads to the algebraic approach, but extended with a proper manipulation of the domain (see below for some indications and otherwise the proper discussion in "Conquest of the Plane" (2011:221-230)). Hence the infinitesimal doesn't have value for the derivative, but may have value for the understanding of the continuum. For teaching at highschool it is valuable that pupils and students develop a proper number sense and understand reports from the calculator. Mathematics needs a good theory of number while there still seems to exist some confusion, in particular in education. (Pure theorists often focus on the Dedekind cut but highschool uses decimals.)

(b) Non-standard analysis revives the infinitesimal. Part of the intuition seems correct: both for the number sense and the dislike of the Weierstraß method. However, non-standard analysis apparently still confluates didactic issues on the number line with didactic issues on the derivative, and this needlessly complicates didactics. This combination makes papers on non-standard analysis hard reading too. Their neglect of the algebraic approach is unwarranted.

The combination of the various results cause a program "Neoclassical mathematics for the schools" (2011).

To emphasize the tentative character of this paper the best introduction is the autobiographical route how I got here.

1.2 ALOE, EWS and COTP

1.2.1 A base in logic

Colignatus (1981, 2007, 2011) *A Logic of Exceptions* (ALOE) was written in 1980-1981, shelved for 25 years, and retyped and published in 2007, with now a 'second edition' in 2011. ALOE solves the Liar paradox, Russell's set paradox and Gödel's misconception with respect to the Gödeliar. It uses three-valued logic and solves the paradoxes of three-valued logic.

I have always been sensitive to history. Originally I intended to study archeology but the horror of Biafra caused me to switch to econometrics to solve such world problems - and then see DRGTPE (2011a). A good book in history will still draw my attention and in this manner ALOE contains many references to the history of logic. It was only because of reading Bochenski (1956, 1970) that I came across Paul of Venice and an approach to resolve Russell's paradox.

Russell's set is $R \equiv \{y \mid y \notin y\}$. This definition can be diagnosed as self-contradictory, whence it is decided that the concept is nonsensical. Using a three-valued logic, the definition is still allowed, i.e. not excluded by a Theory of Types, but statements using it receive a truthvalue Indeterminate. An example of a set similar to Russell's set but without contradiction is the set $S = \{y \mid y \notin y \wedge y \in S\}$, which definition uses a small consistency condition, taken from Paul of Venice, see ALOE p127-129.

PM. Merely writing $S = \{y \neq S \mid y \notin y\}$ might convey the impression that this is a mere choice while it isn't. The notation $S = \{y \mid y \notin y \wedge y \in S\}$ clarifies that $S \notin S$ derives from logic and is not an arbitrary choice. As explained in ALOE:129 the notation $S = \{y \mid y \notin y \wedge y \in S\}$ has the disadvantage that $y \in S \Leftrightarrow (y \notin y \wedge y \in S)$ or $a \Leftrightarrow (b \wedge a)$ which puristically leaves $y \in S$ in the air (or infinite regress). As explained in ALOE:129 a more complex expression is $S = \{y \mid ((y \neq S) \Rightarrow (y \notin y)) \vee ((y = S) \Rightarrow (y \notin y \wedge y \in S))\}$. The expression with the consistency criterion seems adequate as a shorthand for the complexer full statement.

1.2.2 Cantor's Theorem

Using a similar consistency criterion for Cantor's Theorem on the power set we find that its proof collapses. This allows us to speak about a 'set of all sets' (unless we would find some other contradiction). Below we also reject the diagonal argument on the real numbers. ALOE in 2007 still allowed the diagonal argument for the reals only but now in 2011 and its 'second edition' I find that the set of real numbers $\mathbb{R}$ is as large as the set of natural numbers $\mathbb{N}$. My knowledge about Cantor's transfinites is limited to DeLong (1971) and popular discussion like Wallace (2003). Nevertheless it seems possible (see below) to reject the theorem on which they are based - and possibly also their link to infinitesimals.

To be exact: ALOE 2007 still accepted Cantor’s diagonal argument, i.e. the power set theorem only for the infinite set of natural numbers ℕ, so that ℝ was taken as non-denumerable. This acceptance was based upon the exposition in DeLong (1971). However, reading Wallace (2003) in the Summer of 2007 it appeared, in hindsight, that DeLong only presents the diagonal argument sec and does not present a proof of the power set theorem proper. To be exact: DeLong (1971:78) explicitly states that he doesn’t prove Cantor’s theorem and Wallace (2003:275) gives a proof. Had I seen that latter proof before in the context of my logical studies, I would have noted its relation to Russell’s set paradox sooner, and ALOE would have been a bit more guarded, like the ‘second edition’ now is. It may actually be that I had seen the proof in an earlier standard course in mathematics, as we students in econometrics in Groningen had joint courses in mathematics with students in mathematics, physics and astronomy. But then it must not have left much of an impression. What happened in 2007 was that the rejection of Cantor’s general argument got included into ALOE but the diagonal argument was still accepted. I have never been much interested in numbers or the infinite so perhaps I may be excused for this circuitous route. Indeed, my picking up of Wallace (2003) in 2007 is only from a book-sale and the lazy season of the middle of Summer. Which season became a bit less lazy than hoped for.

1.2.3 From ‘paradoxes of division by zero’ towards the continuum

1.2.3.1 Origin in 2007

Retyping ALOE in 2007 also got me to look into the ‘paradoxes of division by zero’. My first step was to distinguish between the noun of static division x / y, and the verb of dynamic division x // y, where the latter also allows algebraic simplification with suitable manipulations of the domain. Once I had formulated that, my thought was: “Oh my, this also reformulates calculus, oh horror, why do I have to see this, this means another section, am I really expected to develop this, and have all this discussion about all those ages of development of calculus ?”.

Elegance with Substance (2009) (EWS) collected and integrated the tidbits that I had observed over the years in the education in mathematics. It presents mathematics not as immutable but rather as a science and an art. Hence I tend to agree with Michael J. Crowe (1988) *Ten Misconceptions about Mathematics and Its History*.

It appeared that EWS could be implemented in a textbook format and this became *Conquest of the Plane* (2011) (COTP) - actually a primer in a textbook format. ALOE p240-242 thus provides a new way to develop the derivative, in three pages on the paradoxes from division by zero. EWS p87-92 extends this explanation that “the derivative is algebra”, and COTP extends it by developing calculus itself. See COTP p48+ and in particular p57 for an extended clarification of the difference between / and //. Thus ALOE, EWS and COTP develop the derivative without the use of limits of infinitesimals.

PM. Discussion about COTP resulted in Colignatus (2011e) *Reading Notes*, of which this present paper has been given a life of itself, intended as a 'mathematical paper'.

1.2.3.2 History via Boyer

COTP however contains much less historical discussion than ALOE. I have now read Boyer (1949) - an excellent study as far as I can judge - to see what might be usefully be included in a future edition. A first conclusion is already that no corrections are required. Boyer (1949:13) also provides us with a fine quote that suggests openness of mind towards new approaches on the derivative:

"Mathematicians now feel that the theory of aggregates has provided the requisite foundations for calculus, for which men had sought since the time of Newton and Leibniz. It is impossible to predict with any confidence, however, that this is the final step in the process of abstracting from the primitive ideas of change and multiplicity all those irrelevant incumbrances with which intuition binds these concepts. It is a natural tendency of man to hypostatize those ideas which have great value for him, but a just appreciation of the origin of the derivative and the integral will make clear how unwarrantedly sanguine is any view which would regard the establishment of these notions as bringing to its ultimate close the development of the concepts of the calculus."

Indeed, Boyer provides some links to the algebraic view of the derivative in ALOE, EWS and COTP:

(a) Boyer p 217-218 quotes Leibniz explaining to Bayle in a letter of 1687: "the difference is not assumed to be zero until the calculation is purged as far as is possible by legitimate omissions, and reduced to ratios of nonevanescent quantities, and we finally come to the point where we apply our result to the ultimate case". This is exactly what I propose to do, with the added clarity of the manipulation of the domain, and some other subtleties in ALOE, EWS and COTP (best take the latter). Apparently Leibniz also had other explanations but this is the algebraic approach.

Boyer comments: "ostensibly by virtue of the law of continuity. Thus even in the work of Leibniz the idea of a limit was implictly invoked". But this is a nonsequitur. You can develop the derivative with limits but you don't have to. Issues of continuity and limits are different subjects that relate to the construction of $\mathbb{R}$ and its functions. Switching to those is a change of subject. The formula x^2 can be manipulated as formula, whether its domain is $\mathbb{N}$ or $\mathbb{R}$. The crucial point for the derivative is the logic of excluding zero from the domain, then (after algebraic simplification) include it, then restrict the domain to it. The manipulative sequence on the domain $\mathbb{R} \setminus \{0\}$, $\mathbb{R}$, $\{0\}$. If there are issues with the domain due to pathological cases then the pathology requires attention and not the basic definition of the derivative. Mathematicians are trained, when mention is made of the derivative, to immediately think about limits and continuity, but they must untrain themselves to do so (comparable to quitting an addiction).

Also, Bolzano's example of a continuous but nondifferential function (Boyer p 269) falls apart when we see that the derivative is not quite a slope but actually the change in surface. For example $|x|'$ at $x = 0$ is 0. Also, Weierstraß's example of such a function (Boyer p285) can be seen as pathological because of its infinite sum, while its derivative can be determined algebraically, and numerically might be infinite in the sense of vertical rather than non-existent.

(b) Boyer p 236 mentions Landen 1758 *The Residual Analysis.*Apparently Landen missed the manipulation of the domain. Boyer criticizes him that he lacked the notion of limit. This however is the same misconception as discussed under (a).

(c) Boyer p 251, 260 mentions Lagrange's algebraic approach, both in 1772 and 1797. Again there is the criticism on continuity and limit, as discussed under (a). My comment would be that Lagrange's use of the Taylor expansion is needlessly complex, and that a simple definition suffices.

Useful for an understanding of Lagrange's method appear to be Fraser (1987) and Fraser (1989), see also his website, and Schremmer & Schremmer (1989), see also Alain Schremmer's website, also on the teaching experience. The difference between my suggestion of $df/dx = \{\Delta f \,//\, \Delta x$, set $\Delta x = 0\}$ and Lagrange's approach (apart from 90% overlap):

(i) the former manipulates the domain, while the latter doesn't,

(ii) the former is happy with the step on the first derivative, while the latter continues with the Taylor expansion,

(c) the former interpretes with the change in surface, while the latter interpretes with the slope,

(d) Lagrange doesn't have knowledge of the historically later numerical approach and I allow that this might be needed for extension to cases beyond the basic algebraic case.

1.2.3.3 Non-standard analysis

ALOE as a book on logic thus contains two different links to analysis and the derivative. Apparently Gödel's 'results' link up with non-standard analysis. As I explain in ALOE p243: "I find it impossible to say anything about this [non-standard analysis] since I have not studied non-standard analysis and don't have the time to study it." I now have had a few hours to look at it. After a cursory look I think that it is not my cup of tea. Nevertheless, it seems that a clear rejection of infinitesimals might be possible. There seems to be a way to save them with a different notion of limit but why would you do so if the derivative can already be formulated in algebraic manner ? It turns out however that some aspects may be useful to resolve didactic issues on the real number line.

1.3 The continuum as an actual infinite

A friend cleaning up her bookcase in 2011 gave me Aczel (2000) *The mystery of the Aleph.* Kronecker was against Cantor's approach. These accounts however do not develop particular counterarguments to Cantor's Theorem or objections against the paradoxical nature of 'the' proof. There is Kronecker's suggestion that mathematics uses the potential infinite and philosophy the actual infinite. But the continuum in the interval [0, 1] presents an actual infinity of points, and when we study this, why not mathematically ? Perhaps Kronecker was happy with the Dedekind cut, but this at least neglects didactic issues on the real number line, and quite possibly does not really resolve the mathematical issue.

Boyer apparently equates the term “discrete” with $\mathbb{N}$. For me, however, “discrete” is equated with a finite subset of $\mathbb{N}$. Below we will see that there is a bijection ‘in the limit’ between $\mathbb{N}$ and $\mathbb{R}$, so that both sets represent the continuum.

Boyer p293-294 curiously states: “From the definitions of number given above, we see that it is not magnitude which is basic, but order. (...) The derivative and the integral, although still defined as limits of characteristic quotients and sums respectively, have, as a result, ultimately become, through the definition of number and limit, not quantitative but ordinal concepts.” Thus is repeated by p 298 “How startling apropos, with respect to the development of the calculus, is the Pythagorean dictum: *All is number !*” And the conclusion in 309: “The history of the concepts of the calculus shows that the explanation of the qualitative is to be made through the quantitative, and the latter is in turn to be explained through the ordinal, perhaps the most fundamental notion in mathematics.”

This type of reasoning is curious since ‘magnitude’ could be equated to ‘order *ad infinitum*’, and hence there still is an essential difference between the finite and the continuum.

Boyer’s focus on continuum and limits derives from the historical focus on them, i.e. the development towards Weierstraß as the lastest stage, and from the neglect of the potentials of the algebraic approach. My proposal is a revival of the algebraic stage. The derivative and integral are algebraic concepts, and only become quantitative when numerical values are assigned to them.

1.4 Hence, groundwork for others

The literature on number theory and the infinite is huge and that my knowledge is limited to only a few pages (that summarize some points of that huge literature, however). My only angle for this present paper is the insight provided in ALOE on some logical relationships, plus two new books of mine since 2007 that focus on mathematics and its education. Given the existence of that huge literature still unknown to me I thus have my hesitations about expressing my thoughts on this subject. When I read those summaries then it might be considered valid however that I do so, since in essence I only express this logical angle.

With the books ALOE, EWS and COTP I feel at ease since I developed the subjects. This present paper is more tentative since I only looked cursorily at number theory and non-standard analysis. The aim of this paper thus is to express these insights such that possibly a person more at home in number theory and the infinity and non-standard analysis reconsiders that latter literature while taking along the insights in ALOE, EWS and COTP.

We use $\Theta = 2\,\pi$.

2. Cantor’s Theorem in general

See ALOE p238-240 for the context.

Cantor’s Theorem holds that there is no bijection between a set and its power set (the set of all its subsets). For finite sets this is easy to show (by mathematical induction). The problem now is for infinite set A such as the natural or real numbers. The proof (in Wallace (2003:275)) is as follows. Let f: $A \to 2^A$ be the hypothetical bijection between (vaguely defined ‘infinite’) A and its power set. Let $\Phi = \{x \in A \mid x \notin f[x]\}$. Clearly Φ is a subset of A and thus there is a $\varphi = f^{-1}[\Phi]$ so that $f[\varphi] = \Phi$. The question now arises whether $\varphi \in \Phi$ itself. We find that $\varphi \in \Phi \Leftrightarrow \varphi \notin f[\varphi] \Leftrightarrow \varphi \notin \Phi$ which is a contradiction. Ergo, there is no such f. This completes the current proof of Cantor’s Theorem. The subsequent discussion is to show that this proof cannot be accepted.

In the same line of reasoning as with Russell’s set paradox, we might hold that above Φ is badly defined since it is self-contradictory under the hypothesis that there is a bijection. A badly defined set cannot be a subset of something. A test on this line of reasoning is to insert the similar small consistency condition, $\Phi = \{x \in A \mid x \notin f[x] \land x \in \Phi\}$. It will be useful to reserve the term Φ for the latter and use Φ’ for the former inconsistent definition. Now we conclude that $\varphi \notin \Phi$ since it cannot satisfy the condition for membership, i.e. we get $\varphi \in \Phi \Leftrightarrow (\varphi \notin f[\varphi] \land \varphi \in \Phi) \Leftrightarrow (\varphi \notin \Phi \land \varphi \in \Phi) \Leftrightarrow$ *falsum*. Puristically speaking, the earlier defined Φ’ differs lexically from the later defined Φ, the first expression being nonsensical and the latter consistent. Φ’ refers to the lexical description but not meaningfully to a set. Using this, we can also use $\Phi^* = \Phi \bigcup \{\varphi\}$ and we can express consistently that $\varphi \in \Phi^*$. So the earlier ‘proof’ above can be seen as using a confused mixture of Φ and Φ^*. (And, puristically, the same “PM” applies as in §1.2.1 on the Russell paradox, so that a puristically proper form is $\Phi = \{x \in A \land\ x \neq f^{-1}[\Phi] \mid x \notin f[x]\}$ but now with the explanation why $f^{-1}[\Phi] \notin \Phi$.)

It follows:

1. that the current proof for Cantor’s Theorem is based upon a badly defined and inherently paradoxical construct, and that the proof evaporates once a sound construct is used.
2. that the theorem is still unproven for (vaguely defined) infinite sets (that is, I am not aware of other proofs). We could call it “Cantor’s Impression” rather than “Cantor’s Conjecture” since Cantor might not have conjectured it if he had been aware of above rejection.
3. that it becomes feasible to speak again about the ‘set of all sets’. This has the advantage that we do not need to distinguish ‘any’ versus ‘all’ sets. And neither between sets versus classes.
4. that the transfinites that are defined by using Cantor’s Theorem evaporate with

it.

5. that the distinction between ℕ and ℝ rests (only) upon the specific diagonal argument (that differs from the general proof) (and it will be discussed below).

When we consider the diagonal argument on ℝ then it appears that we may reject it.

Elwes (2011) reports on a structure on infinity named “Ultimate L”. From the above it seems that it would be based upon a misunderstanding and an unwarranted addiction to two-valued logic instead of the rational choice for three-valued logic. Wu (2011) reports on the mis-education of mathematics teachers and I agree that professional mathematicians should refocus on education.

3. Cantor’s diagonal argument for the real numbers

3.1 Potential and actual infinite

Aristotle’s distinction between the potential and the actual infinite is a superb common sense observation on the workings of the human mind. Elements of ℕ and the notion of repetition or recursion allow us to develop the potential infinite. The actual infinite is developed (a) via abstraction with associated ‘naming’ or (b) the notion of continuity of space (rather than time), or intervals in ℝ. While we use the symbol ℕ to denote the natural numbers, this not merely means that we can give a program to construct integer values consecutively but at the same moment our mind leaps to the idea of the completed whole (represented by the symbol ℕ or the phrase “natural numbers”), even though the latter seems as much a figment of the imagination as the idea of an infinite line. The notion of continuity however for say the interval [0, 1] would be a close encounter with the actual infinite. In the same way it is OK to use the mathematical construct that the decimal expansion of Θ has an infinity of digits, which is apparently the conclusion when we use such decimals. The mathematical notion of a limit expresses the leap from the potential to the actual, though its use and precise definition also appears to depend upon context.

But it is quite another thing to go from these considerations to conclusions on ‘transfinites’. I wholeheartedly agree with Cantor’s plea for freedom but mathematics turns to philosophy indeed if there is no *necessary* reason to distinguish different cardinalities for ℕ and ℝ. See also Edwards (1988) and (2008). If there is no necessity, then Occam’s razor applies. Let us see whether there is necessity.

3.2 The diagonal argument

3.2.1 Restatement

Cantor’s diagonal argument on the non-denumerability of the reals ℝ is presented in DeLong (1971:75&83) and Wallace (2003:254). We assume familiarity with it and quickly restate it. It suffices to assume a bijection between ℕ and ℝ that uses digits $d_{i,j}$: $(1 \sim 0.d_{1,1}d_{1,2}....)$, $(2 \sim 0.d_{2,1}d_{2,2}...)$ ... etcetera. The diagonal number is $n_D = 0.d_{1,1}d_{2,2}....$ taken from that list. A real number that cannot be in the list is $n_C = 0.n_{C,1}n_{C,2}....$ where $n_{C,i} = 2$ iff $d_{i,i} = 1$, and $n_{C,i} = 1$ iff $d_{i,i} \neq 1$. For, if C would be the position in the list then $n_{C,C} = d_{C,C}$ by definition of the list and $n_{C,C} \neq d_{C,C}$ by definition of n_C, which is a contradiction. Nevertheless, n_C would be a true real number and thus should be in the list. (QED). PM. We can create an infinity of such points along the diagonal.

I've seen this argument in 1980 and considered it at some length, and have done so now again. In 2007 I still accepted it. With some more maturity I can better appreciate some 'constructivist' views. One may observe that neither DeLong (1971) nor Wallace (2003) mentions those constructivist considerations on this proof. It would be better if those would be mentioned in summary statements since they better clarify what is at issue. Curiously though I have not found a direct counterargument yet, neither in papers on Kronecker.

PM. This argument generally attracts attention since there is something fishy about taking an element $d_{C,C}$ and redefine it to have another value than it already has.

Note that Cantor does not specify a specific number C for the diagonal digit $d_{C,C}$. His reasoning is non-constructive in the sense that the number cannot be calculated. This might be clarified by writing $C = \infty$ so that we are discussing $n_{\infty,\infty}$ which may be recognized as rather awkward since the symbol ∞ generally stands for "undefined". In that respect it is somewhat curious to allow Cantor this freedom to be nonconstructive while requiring for the below "bijection by abstraction" that an index for 1/3 should be specified while the abstraction causes that this cannot be done.

Cantor's argument is an application of logic that if there is a diagonal then there is a C, yet by abstraction the notion of a diagonal is not defined. Let us become a bit more formal. Let the proposition be p = "There is a (well-formed) diagonal". Cantor suggests the following scheme: $p \Rightarrow \neg p$ ergo $\neg p$. In the abstraction below we will however see that the diagonal is not well-defined so that the true form rather is $\neg p \Rightarrow$ (Cantor's $p \Rightarrow \neg p$), which is an instance of the "ex falso sequitur quodlibet" $\neg p \Rightarrow (p \Rightarrow q)$ with $q = \neg p$. In other words, Cantor implicitly uses that the diagonal does not exist to prove its nonexistence, which is begging the question. This point becomes clearer when we look at the actual construction of ℝ where we will see that the issue of a diagonal is not a well-defined question. That the notion of a diagonal is not well-defined does not prove that ℝ is non-denumerable.

3.2.2 Definition of ℝ

The main point resides with how we define "real numbers". Let us actually define the real numbers and proceed from there. It suffices to look at the points in [0, 1] (and others could be found by $1 / x$ etcetera). First, let d be the number of digits:

For $d = 1$, we have 0.0, 0.1, 0.2, ..., 0.9, 1.0.

For $d = 2$, we have 0.00, 0.01, 0.02, ...0.09, 0.10, 0.11, 0.12, ..., 0.98, 0.99, 1.00.

For $d = 3$, we have 0.000, 0.001, ..., 1.000

Etcetera. Values in ℕ can be assigned to these using this algorithm: For $d = 1$ we assign numbers 0, ... , 10. For $d = 2$ we find that 0 = 0.0 = 0.00 and thus we assign 11 to 0.01, 12 to 0.02, etcetera, skipping 0.10, 0.20, 0.30, ... since those have already been assigned too. Thus the rule is that an assignment of 0 does not require a new number from ℕ. Thus for real numbers with a finite number of digits d in ℝ we associate a finite list of $1 + 10^d$ numbers in ℕ.

Subsequently, we let $d \to \infty$. This creates both ℝ and a map between that ℝ and ℕ.

Many mathematicians seem to regard this construction of ℝ as inferior in some sense and they adopt the Dedekind cut. The decimal construction of ℝ rather is an existence proof and the Dedekind cut an abstraction at a later phase of theory. For the development of the number sense we must focus on the decimals anyway so we need a sound development anyway. Indeed, Gowers (2003) adopts the decimals and shows the creation of the complete ordered field on ℝ. Leviatan (2004) shows its place in didactics. Similarly Schremmer & Schremmer (1989), and see Schremmer (2011) for the notation 1/3 = 0.33 + [...] as a stepping stone for Landau's o, though see COTP §2.2 on approximation and rounding off. It is important that the approximation is recorded in the number and not in the abuse of the equality sign (since ≈ is ambiguous).

PM. This definition of ℝ generates the same result as Gowers (2003). He proceeds as follows: "To begin with, one defines an infinite decimal in the obvious way, as a finite sequence of elements of the set {0,1,2,3,4,5,6,7,8,9} followed by a decimal point followed by an infinite sequence of elements of the set {0,1,2,3,4,5,6,7,8,9}. This isn't quite the whole definition since one must point out that some of these objects are equal: for example, 0124.383478... is the same number as 124.383478... (assuming of course that the sequences continue in the same way) and 1.9999999... is the same number as 2. (About this last example, by the way, there can be no argument, since I am giving a *definition.* I can do this in whatever way I please, and it pleases me to stipulate that 1.999999... = 2 and to make similar stipulations whenever I have an infinite string of nines.) That defines the set I am constructing. To make it a complete ordered field, I must now specify the ordering, explain how to add and multiply infinite decimals, and prove that the field axioms, order axioms and completeness axiom are all satisfied."

3.2.3 Bijection in the limit

Instead of speaking about a bijection it is better to speak about a 'bijection in the limit', if that helps to resolve the paradox and confusion. This is an insight that I do not see in the literature. Perhaps a better term is 'bijection by abstraction' since there are already definitions for Weierstraß limit or Cauchy limit, but the use of the term 'limit' is also warranted since it is the same overall domain of discussion.

Definition: A bijection $b[n] \to b$ can be said to exist in the limit between domain D and range R, if for each n it is a bijection that $b[n]\colon D[n] \to R[n]$, and D and R are the limit values of these, such that $D = \text{limit}[n \to \infty, D[n]]$ and $R = \text{limit}[n \to \infty, R[n]]$.

With this definition, ℕ and ℝ have the same cardinal number, ℕ ~ ℝ, in the (new) sense that there exists a bijection in the limit. It simply defines away Cantor's problem. The potential infinite is the process on 0, 1, 2, ... and the actual infinite is the set ℕ and the intervals in ℝ.

The intention of these terms is to only *capture* what we have been doing in mathematics for ages. It is not intended to present something horribly new. It only describes what we have been doing and what has not been described in these terms before. It is a new photograph but at higher resolution, and it allows to see where Cantor was too quick.

First some general comments and then a reaction to possible misunderstandings concerning the use of the term 'limit'.

3.2.3.1 On the interpretation of 'bijection in the limit'

These are the two observations on the 'to' and 'from' relations:

(a) Above scheme allows for each element in ℕ to determine what number in ℝ is associated with it (and it will have a finite number of digits). The main question is what happens with real numbers with infinite numbers of digits, like 1 / Θ or a simple number like 0.101010... or a truly random sequence. Since we took $d \to \infty$, all such numbers are included in the list. Our construction apparently is valid for the creation of ℝ. Since we have a map to ℕ for each value of *d,* we find ourselves forced to the conclusion that with the creation of ℝ there is *simultaneously* the creation of a map between ℝ and ℕ.

(b) The statement "$d \to \infty$" appears to be vague and insufficiently constructive to the effect we cannot pinpoint a particular value in ℕ associated with say 1 / Θ. It is paradoxical that we can decode a value in ℕ to a particular number in ℝ but that we cannot specify an algorithm to decode from Θ to a particular value in ℕ. The construction with $d \to \infty$ apparently introduces vagueness, even though we can infer that such a map *must have been created* since also ℝ has been created. Actually, it is this very vagueness that causes that we have to distinguish between ℕ and ℝ, and make the distinction between counting and measuring.

(A nonconstructive example is that when someone climbs a mountain between 10 and 12 hours on one day, and descends between 10 and 12 the other day, that there must be at least one moment when he or she is at the same height at the same time. We know this without being able to pinpoint the moment exactly.)

This might also be summarized in this manner. Though the name ℕ suggests an actual infinite, and though the collection is an actual infinite, the natural numbers are rather associated with *counting* and counting is always the potential infinite. Whence ℝ associates much better with the actual infinite given by the totality of ℕ, which is the continuum, which is *measuring*. If you look for something in a filing cabinet or encyclopedia, you might start with A, and step through all values, but it is smarter ('measuring') to jump to the appropriate first letter, etcetera.

An unrepenting constructivist might want to see a constructive bijection between ℕ and ℝ and might reject the vagueness of the 'bijection in the limit'. An eclectic and unrepenting Aristotelian might be happy that both sets have the same ‘cardinal number’, namely infinity, and that there is no necessity for ‘transfinites’.

3.2.3.2 A misunderstanding due to the standard definition of convergence of sets

There is a standard definition of convergence of sets (SDCS) in set theory. This is that points which are infinitely often in the sets of the sequence are in the limit set, and that points which are infinitely often not in the sets are not in the limit.

Since $\mathbb{R}[d]$ with finite d does not contain the irrational numbers, the above definition would only generate the following: $\mathbb{R}[\mathrm{d}] = \mathbb{Q}[d]$, and $d \to \infty$ implies $\mathbb{Q}[d] \to \mathbb{Q}$ (with $\to$ the SDCS).

The problem with this is that the construction of $\mathbb{R}[d]$ is such that all combinations of digits are included, so that we actually have ($\mathbb{R}[\mathrm{d}] = \mathbb{Q}[d]$) $\to \mathbb{R}$, apparently in another sense than SDCS.

Hence the SDCS is inadequate to describe this phenomenon.

(In a discussion, someone accepted that $\mathbb{R}$ was created on the left, indeed, but he rejected that $\mathbb{N}$ came about on the right, since it ‘ought’ to be $2^{\mathbb{N}}$, even though the only step taken was the simultaneous step of taking d to the countable infinite.)

3.2.3.3 A misunderstanding due to ‘replacement’

One topic of discussion might be that we see the step from finite d to infinity as a ‘mere’ replacement of d by the symbol ∞. This could be a form of algebra. It might be relevant for how our actual brains work. It might be relevant for didactics, to suggest to some students who have difficulty understanding what is happening. However, at this point in the discussion there is no developed algebra on such methods, and the proper interpretation still is, only, the switch from the potential infinite to the actual infinite, which is a conceptual jump.

To clarify the issue, the argument can also be presented without the term ‘infinity’.

(1) Potential form: $\mathbb{N}[n] = \{0, 1, 2, ..., n\}$

(2) Actual form $\mathbb{N} = \{0, 1, 2, ...\}$

(3) For every n in $\mathbb{N}$ we have $\mathbb{N}[n]$ @ $\mathbb{N}$

PM. The @ can be read as ‘abstraction’. It records that (1) and (2) are linked in their related concepts and notations. In the potential form for each n there is an n+1, in the actual form there is a conceptual switch. The switch can be interpreted as the change from counting to measuring. Thus instead of ‘bijection in the limit’ we can also use the term ‘bijection by abstraction’.

(4) $\mathbb{R}$ is the set of numbers from 0 and 1. (A number between 0 and 1 is infinite sequence of digits not ending with only 9’s; if it ends with only 0’s we call it terminating. This definition can be done without relying on the SDCS.)

(5) We construct the bijection $b[d]$: $\mathbb{R}[d] \leftrightarrow \mathbb{N}[10^d]$ for d a finite depth of digits.

(6) Definition what it means to have a 'bijection in the limit': This means that

(a) there is a bijection $b[d]$ for domain D and range R, $b[d]$: $D[d] \leftrightarrow R[d]$,

(b) $D[d]$ @ D

(c) $R[d]$ @ R

Bijection in the limit can be denoted b: $D \leftrightarrow R$ or $D \sim R$. In that sense D and R are equally large. When (a) - (c) are satisfied then this is also accepted as sufficient proof that there is a b even though that b no longer needs to be constructive.

(7) Then we get the scheme:

$b[d]$: $\mathbb{R}[d] \leftrightarrow \mathbb{N}[10^d]$

make the conceptual switch on the left, $\mathbb{R}[d]$ @ $\mathbb{R}$, and then simultaneously on the right, $\mathbb{N}[m]$ @ $\mathbb{N}$, for $m = 10^d$:

?: $\mathbb{R}$?? $\mathbb{N}$

Check that indeed $\mathbb{R}$ arises: no holes.

(8) Hence: there is a bijection in the limit between $\mathbb{R}$ and $\mathbb{N}$.

PM. The symbols @ and ~ have been introduced directly and perhaps we can work towards some rules on those, such that we can assume those rules and some weaker property to arrive at the same outcome. Some rules seem to be:

((A @ B) & ($A \sim C$)) $\Rightarrow$ (C @ B) applied to ($\mathbb{R}[d]$ @ $\mathbb{R}$) & ($\mathbb{R}[d] \sim \mathbb{N}[m]$ for some $m = 10^d$) $\Rightarrow$ ($\mathbb{N}[m]$ @ $\mathbb{R}$)

((A @ B) & (A @ C)) $\Rightarrow$ ($B \sim C$) applied to ($\mathbb{N}[m]$ @ $\mathbb{N}$) & ($\mathbb{N}[m]$ @ $\mathbb{R}$) $\Rightarrow$ ($\mathbb{N} \sim \mathbb{R}$)

3.2.3.4 A misunderstanding due to changing the definition of @

One reader argued:

(1) $\mathbb{N}[d]$ @ $\mathbb{N}$ means that for every $n \in \mathbb{N}$ there is an m such that for $d > m$ we have $n \in \mathbb{N}[d]$.

(2) Then $\mathbb{R}[d]$ @ $\mathbb{R}$ means that for every $r \in \mathbb{R}$ there is an m such that for $d > m$ we have $r \in \mathbb{R}[d]$.

(3) The latter however is not true.

(4) Hence the meaning of $a[d]$ @ a differs for $\mathbb{N}$ and $\mathbb{R}$ and thus is not well defined.

In reply: Above, the symbol @ is <u>*not*</u> presented in a general format $a[d]$ @ a. Only the expressions $\mathbb{N}[d]$ @ $\mathbb{N}$ and $\mathbb{R}[d]$ @ $\mathbb{R}$ are defined separately, where it thus matters whether we look at $\mathbb{N}$ or $\mathbb{R}$. The observation by the reader thus is partly accurate since there is indeed no general definition given for $a[d]$ @ a, but it is inaccurate since it wants to impose such a definition while it hasn't been given and it wants to read more than has been written.

3.2.3.5 In sum

Here ends the formal deduction. The interpretation is:

(i) The decimals in [0, 1] can be constructed via a loop on d, the depth of decimals, and then the infinite application using *countable* infinity. This is not radically novel. The distinction between potential and actual infinity is given by Aristotle, and everyone has been aware of a sense of paradox.

(ii) Due to Cantor people have started thinking that the loop would require 'higher' infinity. Cantor's arguments however collapse in three-valued logic (and his universe has strange beasts anyway).

(iii) The concept of 'bijection in the limit' or 'bijection by abstraction' helps to get our feet on the ground again. The potential infinite can be associated with counting and the actual infinite can be associated with the continuum. Clarity restored.

(iv) The clarity actually arises by taking the paradox of the relation between the natural numbers and the continuum as the definition of 'bijection via abstraction'. (The paradox is that for each d we have 10^d decimal numbers but for $d \to \infty$ we lose identification.)

(v) To avoid confusion in discussion: $\mathbb{N}$ is "countably infinite" in all approaches, also via abstraction. $\mathbb{R}$ is "uncountably infinite" in Cantor's view but "countably infinite by abstraction" according to this paper. For $\mathbb{N}$ we might drop the "via abstraction" but for $\mathbb{R}$ we might include it for clarity. We may also say that $\mathbb{R}$ is "Cantor uncountably infinite" for clarity.

3.2.4 The fallacy of composition

When we consider a real value with an infinite number of digits, like Θ or a simple number like 0.101010... or a truly random sequence, we employ the notion of the actual infinite. Instead, with above definition and construction of $\mathbb{R}$ we employ the potential infinite. When we combine these notions then we make the *fallacy of composition.*

It is not quite proper to ask for the value in $\mathbb{N}$ for Θ in the list generated for $\mathbb{R}$, if Θ is still in the process of being built up as an element in $\mathbb{R}$. By taking the limit we get $\mathbb{R}$, including Θ, but taking the limit apparently also means that we resign constructive clarity.

Stating "$d \to \infty$" means a 'leap of faith' or rather a *shift of perspective* from the potential to the actual infinite. Rather than counting 1, 2, 3, we shift to the set of natural numbers, $\mathbb{N}$, and the name "the natural numbers" refers to that actual infinite. When we use that symbol then this does not mean that we actually have a full list of all the natural numbers. We only have the name. The shift in perspective is not per se 'constructive'.

3.2.5 Whence the rejection of the diagonal argument

What about Cantor's diagonal argument ? Apparently it suffers from that fallacy of composition. The list of numbers in $\mathbb{R}$ is created in the manner of a potential infinite but the diagonal proof suggests that they can be accessed as actual infinites.

Above, for each d in the list we might try to take a diagonal but the numbers are not long enough. For $d = 2$ we already get stuck at 0.01. Supposedly though we could extend the numbers with a sufficient length of zero's. Creating a new number based upon such a diagonal number would not be proper since it conflicts with the situation defined for that particular value of d. If we let $d \to \infty$ then there is no well-defined manner to take a diagonal.

At this point, you might already want to look at the table in §3.2.7 below that summarizes the points of view.

Cantor's argument has this structure: "Suppose that there is a list, then there is a diagonal, then a new number is created that cannot be on the diagonal. Hence there is no such list, hence real numbers are not denumerable." But the above showed that there must be a list, that comes about alongside with the creation of $\mathbb{R}$ itself. The alternative conclusion with respect to Cantor's argument is that it is improper to use a 'diagonal' since it is not well-defined and does not exist. The mutation rule on the 'diagonal' is rather a waiting rule than a number creation rule. The numbers are in the list at some point, and do not have to be created anew. We only have to go from one value of d to another value of d to let the mutated number appear (up to the required value of d). For example, if $d = 2$ and the 'diagonal' stops at 0.01, and the mutated number becomes 0.12 then we move up the list and find it somewhere.

The unrepenting constructivist (a third point of view in this discussion) who rejects the usefulness of the 'bijection in the limit' and who wants to see a constructive bijection such that we can calculate the proper number for $1 / \Theta$, would still stick to a constructive approach for the diagonal. Cantor's proof assumes a diagonal but rather that diagonal would be created. While it is constructed, at the same time the mapped value of the diagonal is created, and then it appears that it could not be created since it is inconsistent that $n_{C,C} = d_{C,C}$ by definition of the list and $n_{C,C} \neq d_{C,C}$ by definition of n_C.

3.2.6 Cantor's original argument of 1874

The syllabus on set theory by Hart (2011) opens on page 1 with Cantor's original argument on nondenumerability, which argument he later improved upon with the diagonal argument.

The original argument of 1874 suffers the same fallacy of composition. The formulation of the theorem assumes that $\mathbb{R}$ is built up in the manner of a potential infinite, but the proof uses that all elements are actual infinites. Instead, the proof can only use numbers up to a certain digital depth d, and create the full construction only alongside the construction of $\mathbb{R}$ itself.

(When the limit of d is taken both for $\mathbb{R}[d]$ and the associated intervals $\alpha[d]$ and $\beta[d]$ then there arise curious questions. Regard for example the series with limits α =

0.9999... and $\beta = 1.000...$. It is common to conclude that $\alpha = \beta$ so that there is no η inbetween. See also the below.)

See the Appendix for a longer discussion to show where Cantor went wrong.

3.2.7 A summary of the differences

Given the onslaught since 1874 (if not earlier with Zeno's paradoxes) it may be useful to put the different approaches in a table.

Topic	*Cantor*	*ALOE, EWS* and *COTP* (*Occam*)
Logic	two – valued	three – valued
Cantor's Theorem	Accept	Reject, like with Russell's paradox
Potential & Actual ∞	Commits the fallacy of composition	Proper distinction
Diagonal	Assumption causes rejection	Is not defined in potential form
Mutation rule	Creates a new number	Waiting rule
Bijection	Impossible to create	In the limit
Cardinality	$\mathbb{N} < \mathbb{R}$	$\mathbb{N} \sim \mathbb{R}$

In the latter view the following statements mean precisely the same: (i) the shift in perspective from potential infinity to actual infinity (other than a mere name: thus the continuum), (ii) the imagination of the continuous interval of [0, 1], (iii) regarding this imagination as a constructive act (for geometry), (iv) accepting this to be what we mean by a bijection in the limit between $\mathbb{N}$ and $\mathbb{R}$, (v) the specification of what we mean by 'taking the limit' in this instance.

3.2.8 Conclusion on the continuum

As holds for evolutionary biology where we tend to forget what 'deep time' is, we may forget for the natural numbers what infinity really means. The googol is 10^100. Let $g[n] = n$^...^n with n times ^. For example $g[2] = 2$^(2^2) = 16. Try g[googol], or apply g a googol times to itself, as in g[... g[googol]...]. These are just small numbers compared to what is possible.

The unrepenting constructivist has a strong position and might actually be right. There might be theoretical advantages to assume a continuity with a higher cardinality than the set of natural numbers. Instead of getting entangled in logical knotts we might also use Occam's razor and assume the same cardinality. Above considerations on 'bijection in the limit' would support the latter.

The main reason to accept the diagonal argument and thus different cardinal numbers for $\mathbb{N}$ and $\mathbb{R}$ is rather not 'mathematical' but 'philosophical'. Kronecker's suggestion to use the actual and potential infinities as the demarcation is not convincing. It is rather on how those are applied. The demarcation remains depending upon necessity. Attributed to Occam: "entia non sunt multiplicanda praeter necessitatem"

4. Non-standard analysis

4.1 Introduction

ALOE, EWS and COTP develop calculus without infinitesimals or limits. Taking a limit is paradoxical since zero is excluded while it is precisely the value of interest. Boyer (1949:236) states that D'Alembert had a limit concept "supposed to be attained instead of a terminus which can be approached as closely as desired". This is not the current concept of the limit, and D'Alembert's approach does not help for the derivative if we don't manipulate the domain.

Consider this age-old issue: Average speed is defined as covered distance divided by lapsed time. For speed at a particular moment in time this definition does not help, since the value of lapsed time is zero and we cannot divide by zero. Thus speed or motion at each moment in time 'do not exist', Parmenides would argue. Rather than taking limits, that still exclude zero, it seems better to see instantaneous speed as a conditional, e.g. when you sling a stone and at one moment release it so that you can determine the (average) value that then shows itself. The latter conditionality of course only holds when you want to continue to think in terms of average speed. Calculus then proceeds by seeing speed as a function of time itself.

In ALOE, EWS and COTP I hadn't any knowledge about non-standard analysis but now in this article I have had a few hours to look at it and perhaps I can say that I now have an infinitesimal amount of knowledge about it. (This is a pun. I mean a very little bit, that might be shown even smaller than I think it is.)

Some of Zeno's paradoxes concern infinite divisions. Space can be divided infinitely (at least conceptually) but can we do so with matter too ? Democritus's notion of the atom somehow was transferred into the 'modern notion of the infinitesimal'. Some think that this 'modern notion' is well-defined. I wonder.

An Euclidean point has no extension or size, only position. Taking a line means a shift in dimension. A plane adds another dimension. These are shifts in perspective and not nessarily issues in arithmetic. A line has an infinity of points, but an infinity of points might still have extension or size zero. Could we define new concepts and develop some arithmetic ?

The term "infinitely small" causes me to think that this is rather undefined. Thus for me $1 / \infty$ is undefined (and likely should not be equated with 0). But some authors seem to think otherwise. Let us try at a definition.

4.2 A possible definition of "infinitesimal"

Above definition of $\mathbb{R}$ in §3.2.2 used numbers. A number has no length. An infinity times zero still allows us to go from $\mathbb{N}$ to $\mathbb{R}$, or $\infty * 0 = 1$, we call this "taking the limit", and we see this as a shift in perspective. This is not arithmetic but one type of limiting process. We can conceive of an other kind of limiting process, not based upon *numbers* but based upon the *lengths* that they represent. This is closely related to preconceptions of the continuum. Following Katz & Katz (2011), C.S. Peirce called the first approach in §3.2.2 "pseudo-continuum" and the latter, that we consider now, the proper "continuum".

In above construction of $\mathbb{R}$ for each d there is a smallest nonzero number that we now call length $\epsilon[d] = 10^{-d}$. A series is 0.1, 0.01, 0.001, ... Each number in that series is a small stretch of length from 0 to that number, such that addition with other lengths eventually gives the unit interval [0, 1]. In general $\epsilon[d] * 10^d = 1$. If we write Limit[$d \to \infty$, 10^d] = ∞ still for number, then we may write LengthLimit[$d \to \infty$, 10^{-d}] = $1 / \infty > 0$ for length, and maintain the form $(1 / \infty) * \infty = 1$. We can call $1 / \infty$ "infinitely small" or "infinitesimal". Possibly we might use labels α and ω but rather keep those available for other uses. Then ∞ suffices as long as the context is clear. If there is a term for "infinitely large" then in this manner it would make some sense to think about the "infinitely small" that would not be quite zero. Crucially, ∞ would not yet be a number in $\mathbb{R}$ and thus $1 / \infty$ not a number belonging to $\mathbb{R}$ either. Is a new kind of number, say a 'process indicator', which generates a space $\mathbb{R} \times \{1 / \infty\}$. If we say that x is infinitesimal we mean that x becomes infinitely small, so that 'is' = 'becomes' for this process, since the idea of a process is its 'becoming'.

A possible notation might be $1 / \infty = 0.00...001$. The ellipsis "..." can be ambiguous since it might indicate a finite list of zero's. Now it would indicate an infinite list or process. There is ambiguity again when $(1 / \infty)^2 = 0.00...001 = 1 / \infty$ again. See below for a longer discussion.

The earlier transformation in §3.2.2 of the potential infinity of $\mathbb{N}$ to the actual infinity of the continuum $\mathbb{R}$ might be seen as based upon a wrong concept of limit, and based rather upon the small stretches of length $\epsilon[d]$ that already presuppose the continuum. From this perspective a number itself has length 0 and an infinity times zero would still amount to zero or $\infty * 0 = 0$. With this new LengthLimit we now cannot take $1 / \infty = 0$ since then we would have $0 = 0 * \infty = (1 / \infty) * \infty = 1$. Thus $1 / \infty$ as a process indicator cannot quite be compared to 0. We thus still apply methods of potential infinity but perhaps some algebra of this type of actual infinity might be developed (as non-standard analysis apparently does).

Boyer (1949:170-171): "Wallis [1656] for his part said that $1 / \infty$ represented an infinitely small quantity, a *non-quanta*. A parallellogram whose altitude is infinitely small or zero is therefor "scarcely anything but a line", except that this line is supposed "extensible, or to have such a small thickness that by an infinite multiplication a certain altitude or width can be acquired". Such thickness indeed suggests another dimension next to $\mathbb{R}$ as defined in §3.2.2.

Robinson's development of non-standard analysis apparently indeed develops a different space $^*\mathbb{R}$ (see Davis & Hersh (1980)). We might also regard a definition along the lines of Weierstraß, and now taking $d \in \mathbb{R}$:

Possible definition: A real number $d > 0$ is infinitesimal iff $\forall\, \epsilon > 0$ with $\forall f$: $\epsilon \neq f[d]$, it holds that $\epsilon \geqslant d > 0$.

The reference to the function f establishes that d and ϵ are independent for otherwise it would be simple to take $\epsilon = d^2$ and show an inconsistency. With this definition it still holds that $d^2 < d$ but this does not invalidate the definition or concept. A possible interpretation is to think of d as a variable on the domain in $\mathbb{R}$ which turns the issue into algebra. The crucial point would be that the variable would not get assigned a particular value because then it would be possible to find a lower point - yet this would use the information about its value and thus violate the condition on the independence. It would still seem simpler to just write $1 / \infty$ instead of d or 0.00...001. Above possible definition on d however helps us to ask whether a process might still not be thought of as belonging to $\mathbb{R}$.

(The interpretation of d as a variable and the switch to algebra, to be used for the derivative, however loses from the algebraic approach in ALOE, EWS and COTP for the derivative. We only discuss the real number line here. While such definitions seem conceivable, it does not strike one as a bedrock foundation for calculus.It still excludes the value of zero that is precisely relevant at the point where the derivative is taken.)

4.3 Where to put the dots ?

In the definition of $\mathbb{R}$ in §3.2.2 we still seem to have some freedom to place dots. Suppose that we take the liberty to write down the 'end-digits' as well. Then $1/\infty$ would not be just a process but it can be included in the list and thus appear part of $\mathbb{R}$. Well, it is a paradoxial subject. On this notation, see also Ely (2010:129-137), Katz & Katz (2009:10-11) (though with 0.00...001 for finite digits and Lightstone's semicolon for infinity) and the discussion in §4.7 below. (PM. The text editor is *Mathematica*, and the table format apparently does not allow to connect the dots since this new number format is not recognized. But 0 = 0.00...00 etcetera.)

$$\begin{array}{cccc}
0 & = & 0.0000\ldots & 00\,000 \\
1/\infty & = & 0.0000\ldots & 00\,001 \\
 & \ldots & & \\
\alpha & = & 0.4999\ldots & 99\,999 \\
\beta & = & 0.5000\ldots & 00\,000 \\
 & \ldots & & \\
\gamma & = & 0.9999\ldots & 99\,999 \\
1 & = & 1.0000\ldots & 00\,000
\end{array}$$

In the standard view $\alpha = \beta$ and $\gamma = 1$. The proof: $10\,\gamma = 9.999...$ hence $(10\,\gamma - \gamma) = (9.999 - \gamma)$ or $9\,\gamma = 9$ or $\gamma = 1$. This assumes that we can do such arithmetic with such numbers that have the ellipsis 0.999.... The advantage is here that 1/3 = 0.999... / 3 = 0.333...

However, if we take the liberty to write the 'end-digits' for still an infinite process then there is a difference of $1 / \infty$. In that case 1/3 cannot be found in above list.

The issue is related to the notion of 'density'. The continuum would be 'dense' to the effect that between each two numbers there is a third number. If we write 0.9999... and

1.000... then there would be no number inbetween and the two strings of digits would represent the same number. Being numbers, both have a measure / size / extension / length of 0, and since these two zero lengths ‘lie next to each other’, their distance must be zero, so that these are the same number. However, this notion of ‘density’ meets with an alternative. If we assume the liberty to write down the ‘end-digits’ then there is a difference of $1 / \infty$. The numbers are densely packed in the sense that each number signifies the end of a length of $1/\infty$ distance from the other number. This approach fits the notion of the LengthLimit. It allows $1 / \infty$ to be included in $\mathbb{R}$ again. It would not be a symbol like Θ or e of which the first digits can be determined. It hence would be of a different type, but still part of $\mathbb{R}$.

With a function $f[x, \epsilon[d]]$ we can imagine a FinalLimit$[1/\infty \to 0, f[x, 1/\infty]] = f[x, 0]$, that eliminates the distance between $1/\infty$ and 0. This merely means the *change in perspective* that there is no number between 0 and 0.00...001 so that these are the same number.

Hence Limit$[x \to 0, f[y, x]]$ = FinalLimit$[1/\infty \to 0$, LengthLimit$[x \to 1/\infty,\ f[y, x]]]$.

An overview (see also the next section) (this still has an omission of another context where ∞ might be 1 / 0, perhaps we really need different symbols):

Occam	Standard	Non – Standard
LengthLimit : $x \to 1 / \infty$	n.a.	$\mathbb{R} \to {}^*\mathbb{R}$
FinalLimit : $1 / \infty \to 0$	n.a.	$\mathrm{st}[dx] = 0$
Limit : $x \to 0$	$x \to 0$	n.a.
Algebra : $\Delta x = 0$	n.a.	n.a.

In this, the choice of $x = 1 / \infty$ would be mere substitution if $1/\infty$ is accepted as element of $\mathbb{R}$. The FinalLimit would be the crucial step. A circle might still be seen as differing from the inscribed infinity of polygons with bases $1 / \infty$, but this difference disappears when $1 / \infty$ is replaced with zero.

All this still does not solve $(1 / \infty)^2 = 0.00...001 = 1 / \infty$, but perhaps some arithmetic can be found.

The use of LenghtLimit does not mean that $1 / \infty$ really ‘exists’. It is a mathematical tool to formulate some thoughts that linger in the human mind when we try to make sense of the relation between numbers and space. Its prime use would be in education to resolve paradoxical notions that hinder understanding.

The notation 0.00...01 might still be necessary from a didactic point of view. Some non-standard analysis authors use a super-lens to focus in on 0, and then they locate the infinitesimal with a mark close to it. This makes only some sense if there is some terminating 1. Otherwise the gimmick is delusional and counterproductive and a complete waste of time.

Zeno’s paradoxes still cause teasing questions. The best we can do now seems to allow for these kinds of limits, as they generate different perspectives that can be useful for some solutions. For example, the derivative first might be formulated with the infinitesimal $1 /\infty$ and subsequently it could be removed by the final limit. It would be a paraphrase of the common definition of the derivative, where the 'discarding' of the infinitesimal amounts to setting it to zero. It would at least allow an important line of development in the history of calculus, including various conceptions of the continuum.

Incidently, the approach to define ℝ first on the rationals and then extend with the irrationals, and then use the Dedekind Cut to define and show continuity (Boyer page 291-292), seems more complicated than §3.2.2, and runs up against said 'rigour, with footnotes' (in this case footnote 77, though perhaps resolved since 1949).

4.4 My understanding of non-standard analysis

In non-standard analysis, ℝ is extended with such infinitesimals into *ℝ (in above perspective it is actually reduced, since you first go to infinitesimals and then to zero). Since infinitesimal $1 / \infty$ is a process and cannot be equated to zero, it cannot be set to zero. But it can be 'discarded'. Thus the procedure st : *ℝ → ℝ takes the standard part of an expression with infinitesimals. For example, the derivative of x^2 can be found as $st[\ 2x + \boldsymbol{dx}\ ] = 2x$. As Fermat already used the notion of "adequality" in 1643 (see Boyer (1949:155-156) and Katz & Katz (2011)) we can use this word for this map, and also write $2x + \boldsymbol{dx} \sim 2x$.

In a sense, it is advisable that non-standard analysis is formulated in a new terminology, to avoid confusion with familiar words and concepts. At the same time it introduces new confusion when the same topics are dealt with. When you go to the bakery to buy a bread, you can say it in English or French, but you are still asking for a bread. The translation is only useful when the environment has changed language, and you only learn the new language to say just the same. To start using "wörlphf" for "table" is not really enlightening.

Non-standard analysis is mainly developed for the derivative. It remains paradoxical with respect to the value $\Delta x = 0$. See Colignatus (2011e), the Reading Notes for COTP, for the diagram what the paradox means: there is no contradiction in the mathematical implemention but there is a contradiction within the conceptual environment that generates that implementation.

Katz & Katz (2011) on the notion of 'adequality': "As far as the logical criticism formulated by Rev. George [Berkeley] is concerned, Fermat's adequality had pre-emptively provided the seeds of an answer, a century before the bishop ever lifted up his pen to write *The Analyst*." Clearly these are only seeds, since Fermat apparently still located $\boldsymbol{dx}$ in ℝ, and his discarding still amounted to setting $\boldsymbol{dx} = 0$, so that Berkeley's question was quite legitimate.

Berkeley's question nowadays would be: why do so difficult with non-standard analysis, when we have the refoundation of the algebraic approach in ALOE, EWS and COTP ?

Davis & Hersh (1980:253): "We have as before $\boldsymbol{ds/dt} = 32 + 16\boldsymbol{dt},$ but now we immediately conclude, rigorously and without any limiting argument, that $v,$ the standard part of $\boldsymbol{ds/dt}$, equals 32." Thus instead of setting $\boldsymbol{dt} = 0$ the term is 'neglected'. You do not see the stain in the carpet if you do not look at it. We may discuss the issue in terms of apples and oranges (where the latter may stand for zero):

(1) Standard analysis says: (a) neglect the orange, (b) peel, cut and core the apple, (c) eat the apple. At least you have had some fruit.

(2) Non-standard analysis says: (a) peel the orange, and take it apart in its parts, (b) peel, cut and core the apple, (c) neglect the orange and eat the apple. At least you have had some fruit.

(3) The algebraic approach is: (a) neglect the apple, (b) peel the orange and take it apart in its parts, (c) eat the orange.

4.5 Lakatos and Cauchy

A friend induced me to read Lakatos (1978a) but, since it wasn't officially published before his death, it must be read in direct conjunction with Dauben (1988) who corrects some basic errors. Lakatos refers to Boyer p273 and discusses Cauchy's term "variable" in conjunction with Cauchy's definition of the infinitesimal as a variable converging to zero but not becoming zero. Cauchy's term "variable" actually means what we now regard as a *sequence.* Cleave (1979) is clearer on this than Lakatos. For example if ϵ is seen as a (modern) variable having values of $1/n$ in the sequence 1/2, 1/3, 1/4, ... then we can indeed imagine ϵ as 'getting smaller and smaller but never becoming zero'. But this 'getting' is better seen as a functional dependency.

As my training has been in the Weierstraß method all this seems a bit arcane to me. Why employ in calculus the baroque construct of an ordered sequence as an object of itself ? I think that there has been progress from Cauchy to Weierstraß, with the modern concept of a variable and taking a limit of $1/n$ with $n \to \infty$. The modern idea of a variable is a label or placeholder that can be assigned an arbitrary value in a domain.

It may well be that there is a shift in perspective: Cauchy looking horizontally and we moderns looking vertically.

$$\begin{array}{cccccc} \textit{variable} & x & 1 & 1/2 & 1/3 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \ldots \\ \textit{function} & f[x] & f[1] & f[1/2] & f[1/3] & \ldots \end{array}$$

Similarly, when we regard the expression Limit[$x \to 1 / \infty$, $f[x]$] then x might be said to be an infinitesimal, as it is the variable that is thought to be subjected to the limiting process. The Cauchy way of thinking in terms of a process thus survives in the notation, even though standard analysis takes that notation to be a summary of the Weierstraß rule. (And the notation Limit[$x \to 0$, $f[x]$] apparently might be an awkward shorthand when 0 can or can't be attained.)

4.6 Others

Borovik & Katz (2011) and Katz & Katz (2011) are very instructive on the issue.

Borovik & Katz (2011) suggest that the infinitesimal can be seen to come about as a jump from a verb to a noun, or from a process to a concept (and they refer to Gray &

Tall 1994 "procept", Dubinsky 1991 "encapsulation" and Sfard 1991 "reification"; while Katz & Katz (2011) also refer to an older philosophical term "hypostatisation"). This seems to be accurate in the sense of identifying such a variable. This does not mean that other aspects are immediately defined. For the shift of focus from the potential to the actual infinity we have a well-defined shift from $\mathbb{N}$ as a process to $\mathbb{R}$ as the continuum. But the rules of 'arithmetic' with respect to infinitesimal $1 / \infty$ are not directly given.

Matt Insall and Eric Weisstein of Math-World clarify that its inifinitesimals are "numbers that are less than 1/2, 1/3, 1/4, 1/5, ..., but greater than 0". This is gibberish to me since I see a sequence and no number. There is no definition what this "less than" would mean if such a number is absent. Katz & Katz (2011) rightly remark that this is better: "Both Carnot and Cauchy say that an infinitesimal is generated by a variable quantity that becomes smaller than any fixed quantity. No contradiction here." This is only clear though once you have learned that this "variable quantity" has another domain than $\mathbb{R}$, for Cauchy a null sequence, and for non-standard analysis $^*\mathbb{R}$, such that the "smaller than" is the complex $\forall \epsilon > 0 : st[\, \boldsymbol{dx} \,] = 0 < \epsilon$. Perhaps it resolves all discussion if we define $1/\infty = 0.00...001$ and accept that we still need the algebraic approach for the derivative.

Curiously Dauben p194 mentions that Cantor "adamently" rejected infinitesimals. Also, Dauben p196: "Kurt Gödel valued Robinson's achievement for similar reasons: it succeeded in uniting mathematics and logic in an essential, fundamental way. That union has proved to be not only one of considerable mathematical importance, but of substantial philosophical and historical content as well." ALOE rejects Gödel's analysis on his incompleteness theorem as essentially misguided. However, the link to non-standard analysis would be based upon his completeness theorem, which is another theorem. Nevertheless, reading Boyer on the winding road for calculus and reading Lakatos on all kinds of consequences of Gödel's analysis that however hangs in the air, I fear that we must be careful with historical issues and quotes from popular authors. This is of course only a fear and not a proper conclusion based upon a deeper study. (I did not even try Keisler (1986, 2010).)

What is interesting though is that there apparently has been an experiment in comparing the results of teaching both Weierstraß and Robinson on the derivative. Hopefully there can be an experiment with the approach in ALOE, EWS and COTP too.

There is always Brouwer and intuitionism. See the chapter in ALOE. Kuiper (2004) seems like a useful recent contribution with clearer reformulations. My impression is that time and space are qualitatively different, and that space is a more natural environment for the hypothesis of continuity. But opinions may differ and it would suffice to discuss what is mathematically necessary.

4.7 Education

Ely (2010) may be read either as the embarrassment of our system of education or as the valiant effort like by the Baron von Münchhausen to save education by lifting itself from its own morass. Ely refers to the education on the real line, where the ambiguity about 0.999... and 1 exists since Stevin and is allowed to continue till our times. EWS clarifies that the chaos holds for many more topics. My advice in EWS to have a parliamentary enquiry into mathematics education can be underlined again, in boldface pointsize 72 with flowers and Chanel 5. But let us look at the real line.

Above I suggested the notation 0.00...001. Subsequently I started a search on the internet whether that notation had been used before. This gave Ely (2010) who shows that a student Sarah invented this notation too (it is a long article, see in particular p129-131 and p137). This fits my idea: in didactics, listen to students. However, this can't resolve fundamental issues in mathematics and here we require good theory. With a good educational plan, Ely could have used the structured survey instead of the open interview. The use of the latter on *numbers* is a testimonium paupertatis. There is also Katz & Katz (2009) who use 0.999...9 but for finite lists, and they refer to Lightstone's semicolon for infinite lists. These are long texts to read while for me the basic question was whether there is an arithmetic for 0.00...001. This is still not clear and I have to put an end to a search somewhere.

Thus, the suggestion is that theorists on non-standard analysis switch to the algebraic approach, and design a didactically clear system for the real number line to be used in education.

4.8 Lakatos on quasi-empirics

EWS holds that mathematicians are trained for abstract thought but that education is an empirical issue. Lakatos (1978b), apparently trained in abstract thought, still was sensitive to Popper's "conjectures and refutations" and falsificationism, and came up with his own "proofs and refutations" and characterisation of mathematics as "quasi-empirical". It is OK to observe that mathematics uses argumentation, with pro's and contra's. But empirical issues only enter in education and application, while empirical experience often generates inspiration for abstraction. Since we already have a word "abstract thought" I do not think that we need another word like "blot-phf" or "quasi-empirical". I mean, there is an urgent need to redesign the education of mathematicians towards sound familiarity with empirical matters, and it would be a great confusion to think that it is already so in sufficient manner.

Lakatos's work incidently appears to be hard reading because he takes Gödel's incompleteness theorems as basic and extends from there. If he had been really empirical then he would have had cause to doubt them, see ALOE. It still would be useful to have a "Lakatos Revisited" book to save the analyses that are still relevant (e.g. his discussion on Newton's legacy).

5. Textbooks with news

ALOE and COTP have been written in textbook format targetted at freshmen. They build up from the basics and include the news along the way, and conclude with discussion at the advanced level. COTP is explicitly a primer for mathematics education (though in textbook format). Thus, these books explicitly differ from standard textbooks that reprocess known material and that do not contain news. The reason for this construction is that it appeared to be the best format to communicate these new results. The steps from the basics onwards apparently can be so subtle and so internally interconnected with the various new insights that it is better to *show* what it amounts to rather than explicitly *discuss* the results separately.

I suggested the two books to a Book List of a general mathematical society. The books were rejected since they had the textbook format and thus apparently were considered not to contain news relevant for mathematicians in that society. I wonder what Boyer would have thought of this, who wrote these few lines on "the formost textbook of modern times" (1950). He selects Euclid's *Elements*, Al-Khwarizmi *Al-jabr*, and Euler's *Introductio in analysin infinitorum*. It is not my intention to compare ALOE and COTP to these books in terms of standing but only with respect to the suggestion that these authors did not merely reprocess known material but also collected original findings of their own to create a coherent whole.

6. Conclusion

A consequence of "A Logic of Exceptions" (ALOE, draft 1981, 2007, 2nd edition 2011) is that it refutes 'the' general proof of Cantor's Theorem (on the power set), so that it only holds for finite sets but not for 'any' set. The diagonal argument on the real numbers can be rejected as well (a new finding now in 2011). There is a bijection 'in the limit' or 'by abstraction' between $\mathbb{N}$ and $\mathbb{R}$. If no contradiction turns up it would become feasible to use the notion of a 'set of all sets' $\mathbb{S}$, as it would no longer be considered a contradiction that the power set of $\mathbb{S}$ would be an element and subset of $\mathbb{S}$ itself. The books ALOE, "Elegance with Substance" (2009) and "Conquest of the Plane" (2011) also develop calculus without the use of limits or infinitesimals. Lagrange's algebraic approach is best supplemented with a manipulation of the domain. Non-standard analysis is not needed for the derivative. Some aspects of it may be reformulated and then may be of use for the education on the real number line. This paper puts these results into historical perspective.

Appendix: Rejection of Cantor's original proof

Taken from Hart (2011):

> Wenn eine nach irgendeinerm Gesetze gegebenen unendliche Reihe von einander verschiedener reeller Zahlgrößen
>
> $$\omega_1, \omega_2, \ldots, \omega_\nu, \ldots \qquad (4)$$
>
> vorliegt, so läßt sich in jedem vorgegebenen Intervalle $(\alpha \ldots \beta)$ eine Zahl η (und folglich unendlich viele solcher Zahlen) bestimmen, welche in der Reihe (4) nicht vorkommt; dies sol nun bewiesen werden. *G. Cantor* [1874]

Unfortunately Hart (2011) uses Dutch so we now use the text from Wikipedia March 6 2012 after checking that it fits with Hart (2011):

> Next Cantor proves his second theorem: Given any sequence of real numbers x_1, *x* interval [*a*, *b*], one can determine a number in [*a*, *b*] that is not contained in the give
>
> To find such a number, Cantor builds two sequences of real numbers as follows: Fi numbers of the given sequence x_1, x_2, x_3, ... that belong to the interior of the interva Designate the smaller of these two numbers by a_1, and the larger by b_1. Similarly, fi numbers of the given sequence belonging to the interior of the interval [a_1, b_1]. Desi by a_2 and the larger by b_2. Continuing this procedure generates a sequence of inte b_2], ... such that each interval in the sequence contains all succeeding intervals. Thi sequence a_1, a_2, a_3, ... is increasing, the sequence b_1, b_2, b_3, ... is decreasing, an the first sequence is smaller than every member of the second sequence.
>
> Cantor now breaks the proof into two cases: Either the number of intervals generat infinite. If finite, let [a_N, b_N] be the last interval. Since at most one x_n can belong to th b_N], any number belonging to the interior besides x_n is not contained in the given se
>
> If the number of intervals is infinite, let $a_\infty = \lim_{n\to\infty} a_n$.[13] At this point, Cantor coul noting that a_∞ is not contained in the given sequence since for every *n*, a_∞ belongs b_n] but x_n does not.[14]
>
> Instead Cantor analyzes the situation further. He lets $b_\infty = \lim_{n\to\infty} b_n$,[15] and then br two cases: $a_\infty = b_\infty$ and $a_\infty < b_\infty$. In the first case, as mentioned above, a_∞ is not co sequence. In the second case, any real number in [a_∞, b_∞] is not contained in the gi Cantor observes that the sequence of real algebraic numbers falls into the first cas how his proof handles this particular sequence.[16]

Let us now redo this method of proof using the $\mathbb{R}[1]$, ... , $\mathbb{R}[d]$. As said the numbers are ranked up to 10^d. For clarity we can take the news $D[d] = \mathbb{R}[d] \setminus \mathbb{R}[d\text{-}1]$, and then rank the digits as $X[d] = D[1] \cup D[2] \cup \ldots \cup D[d] = \{x_0, x_1, \ldots, x_{10^d}\}$, where the union maintains order. Taking the interval from $[a, b]$ generates $[a[d], b[d]]$. For example, if we start on [0, 1] then $[a[1], b[1]]$ = [0.1, 0.2], then [0.11, 0.12], [0.111, 0.112] and so on. (Rather nicely we might think of the limit value of 1/9.)

We now take $\mathbb{R}[d]$ @ $\mathbb{R}$. Subsequently also $X[d]$ @ X. Clearly X is only a permutation of $\mathbb{R}$, and all numbers are represented. Let us denoted the final interval as $[\alpha, \beta]$.

The suggestion that there is an $\eta \in [\alpha, \beta]$ but $\eta \notin \mathbb{R}$ is erroneous since we see that all $\mathbb{R}$ are represented.

Thus there is something crooked in this method of proof. Note that there is no finite number to find the final interval. Note that taking the interior of $[\alpha, \beta]$ is impossible if α

$= \beta$. Taking the interior of $[a[d], b[d]]$ is quite possible since the numbers are defined such that $a[d] \neq b[d]$. But the notion of an "interior" apparently loses meaning when we take the step of abstraction.

(Regard for example the series with limits $\alpha = 0.9999...$ and $\beta = 1.000....$ It is common to conclude that $\alpha = \beta$ so that there is no η inbetween.)

This completes the rejection of Cantor's original proof.

Discussion: His proof *seems* to work since he assumes that $\mathbb{R}$ is built up in the manner of a potential infinite, but the proof uses that all elements have an actual infinity of digits. Instead, the proof can only use numbers up to a certain digital depth *d,* and create the interval only alongside the construction of $\mathbb{R}$ itself. The notion of an interior uses a distance measure that relies on actual infinites, and this apparently also conflicts with the construction of $\mathbb{R}$ from $\mathbb{R}[d]$. Cantor assumes that he can define the various notions independently, but they get only meaning in their mutual dependence, and then must be constructed in a dependent manner.

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Colignatus is the name of Thomas Cool in science.

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