Improving the logical base of calculus on the issue of “division by zero”

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Summary

Define y / x as the “outcome” of division and y // x as the “procedure” of division that will have the outcome y / x if the denominator x cannot be eliminated but that will generate another result when the expression can be simplified algebraically. Using y // x with x possibly becoming zero will not be paradoxical when the paradoxical part has first been eliminated by algebraic simplication. The Weierstra*sz* $\epsilon > 0$ and $\delta > 0$ and its shorthand for the derivative $lim(\Delta x \rightarrow 0)\ \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken. Instead, using Δf // Δx and then setting $\Delta x = 0$ is not paradoxical at all. Much of calculus might well do without the limit idea and it could be advantageous to see calculus as part of algebra rather than a separate subject. This is not just a didactic observation but an essential refoundation of calculus. E.g. the derivative of $| x |$ traditionally is undefined at $x = 0$ but would algebraically be sgn[x], and so on.

Introduction

Since its invention, the zero has been giving trouble. We learned that y / x is undefined when $x = 0$ but precisely that value appears important in calculus and thus an elaborate edifice was created to deal with that. Consider the following four expressions, the first three well-known and the fourth a new design.

1. The differential quotient $\Delta f / \Delta x = (f[x + \Delta x] - f[x]) / \Delta x$ for $\Delta x \neq 0$. Note that one would see this as a result and not as a procedure.
2. The derivative $f'[x] = df / dx = lim_{\Delta x \to 0}\, \Delta f / \Delta x$.
3. The current theoretical true meaning of the derivative with outcome value L: $\forall\, \epsilon > 0\ \exists\, \delta > 0$ so that $|\, \Delta f / \Delta x - L\, | < \epsilon$ for $0 < |\, \Delta x\, | < \delta$.
4. The new suggestion: $f'[x] = \{\Delta f \mathbin{//} \Delta x\ \&\ \Delta x := 0\}$. This means first simplifying the differential quotient and then setting Δx to zero.

Here $y \mathbin{//} x$ is defined as the "procedure of division" that will have the outcome y / x if the denominator x cannot be eliminated and that will generate another result when the expression can be simplified algebraically. The symbols "//" and ":=" express activities or procedures rather than completed results. For example, 4 // 2 would be the activity of dividing 4 by 2, generating the result 1/2. Clearly, 4 // 0 would be undefined just like 4 / 0. But $dx^2 / dx = \{((x + \Delta x)^2 - x^2) \mathbin{//} \Delta x\ \&\ \Delta x := 0\} = \{2x + \Delta x\ \&\ \Delta x := 0\} = 2x$ handles a seeming "division by zero" that actually is no such division. While the Weierstrasz approach in format 3 above uses predicate logic to identify the limit values, the approach in format 4 uses the logic of formula manipulations.

Since we have been taught not to divide without writing down that the denominator ought to be nonzero, the following explanation will help for the proper interpretation of the derivative: first the expression is simplified for $\Delta x <> 0$, then the result is declared valid also for the domain $\Delta x = 0$, and then Δx is set to the value 0. The reason for this declaration of validity resides rather not in the Weierstrasz approach but in the algebraic nature of the elimination of a symbol, as in e.g. $x \mathbin{//} x = 1$.

In a way, this suggested notation is nothing new since it merely codifies what we have been doing since Leibniz and Newton (alphabetical order). In another respect, the approach is a bit different since the discussion of "infinitesimals", i.e. the "quantities vanishing to zero", is avoided while the interpretation given by Weierstrasz and codified in the notion of a limit is rejected since that definition of the limit excludes the value $\Delta x = 0$ which actually is precisely the value of interest at the point where the limit is taken.

Undoubtedly, the notion of the limit (and Weierstrasz's implementation) remain useful for other purposes. It also remains an important tool to vary Δx to show that the derivative indeed gives the slope. That said, the discussion can be simplified and pruned from paradoxes.

It remains to discuss "what to make of this".

Aspect 1: theoretical origin

When we look at the issue from this new algebraic angle, the problem in calculus has not been caused by the "infinitesimals" but by the confusion between "/" and "//". At school, Leibniz, Newton and Weierstrasz were trained to regard y / x as sacrosanct such that it indeed doesn't have a value for $x = 0$. They worked around that, so that algebraically y / x could be simplified before x got its value. While doing so, they created a new math that appeared useful for other realms. These new results gave them confidence that they were on the right track. Yet, they also created something overly complex and essentially inconsistent. Infinitesimals are curious constructs with no coherent meaning. Bishop Berkeley criticized the use of infinitesimals, that were both quantities and zero: who could accept all that need, according to him, "not be squeamish about any point in divinity". The standard story is that Weierstrasz set the record straight. However, Weierstrasz's limit is undefined at precisely the relevant point of interest. "Arbitrary close" is a curious notion for results that seem perfectly exact.

Aspect 2: division in general

We can also reap the benefit from other cases where we supposedly divide by zero. For example, there is no need to be very strict about always writing "//". Once the idea is clear, we might simply keep on writing "/". An expression like $(1 - x^2) / (1 - x)$ would be undefined at $x = 1$ but the natural tendency is to simplify to $1 + x$ and not to include a note that $x \neq 1$ since there is nothing in the context that suggests that we would need to be so pedantic. Standard graphical routines also skip the undefined point (see the graph below). The current teaching and math exam practice is to use the division y / x as a hidden code that must be cracked to find where $x = 0$ but it should rather be the reverse that such undefined points must be explicitly provided if those values are germane to the discussion.

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Simplify[(1 - x^2) / (1 - x)]
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$x + 1$

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Plot[(1 - x^2) / (1 - x), {x, -2, 2}];
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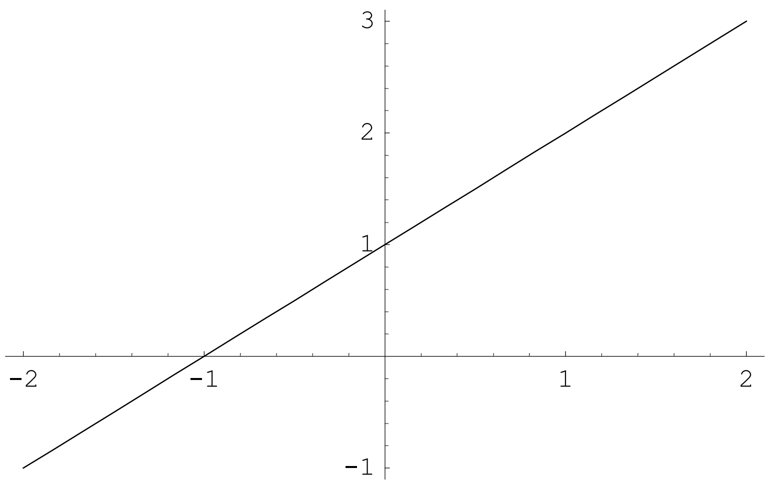

As a note, it is useful to observe the following. The classic example of the inappropriateness of division by zero is the equation $(x - x)\,(x + x) = x^2 - x^2 = (x - x)\,x$, where division by $(x - x)$ causes $x + x = x$ or $2 = 1$. This is also a good example for the clarification that, indeed, one should never divide by zero, so that we must distinguish between:

- creation of a fraction such as putting “/” or “//” between “(x - x) (x + x)” and “(x - x)”
- handling of a fraction such as (x - x) (x + x) (/ or //) (x - x) once it has been created.

The first can be the great Sin that creates such nonsense as 2 = 1, the second is only the application of the rules of algebra. In this case, the algebraic rules tell us that $x - x = 0$, so that simplification generates a value Indeterminate, and this would hold for both / and //. Also a (x + x) / a would generate 2x for $a \neq 0$ and be undefined for $a = 0$. However, if we would have an expression a (x + x) // a then the result would be 2x, and this result would also hold for $a = x - x = 0$.

Aspect 3: students

Generations of students have been suffering. Teachers of math seem to have overcome their own difficulties and thereafter don’t seem to notice the inherent inconsistencies.

Conversely, if the new notation is adopted then the algebraic origin of the derivative will be recognized, strengthening the insights in logic and algebra. Time can be won for more relevant issues.

Teachers may be less tempted to distinguish between “those who know the truth” (the ϵ and δ) (who thus actually are wrongfooted) and “those who only learn the tricks” (unadorned calculus).

Aspect 4: integration

With $f'[x]$ the derivative of $f[x]$ it is straightforward to say that the latter is the integral of the former (i.e. assign the label “integral” to the reverse operation). It is less obvious that such an integral concerns the surface under the curve. But this demonstration could be done in reverse order, by starting with a curve that gives the surface and then find the derivative. Let $F[x]$ be the surface under $y = f[x]$ till x,then the increase in surface is

$$\Delta F = F[x + \Delta x] - F[x] = \Delta x\, (y + \Delta y/2) \qquad (*)$$

and we find $y = f[x] = F'[x]$ as usual. Note in this derivation that it is tempting to divide both sides by Δx but of course we cannot do so directly if Δx is to be set at zero. What we can do however is use the definition of the derivative and substitute the surface change, i.e. $F'[x] = \{\Delta F \;//\; \Delta x \;\&\; \Delta x = 0\} = \{\Delta x\,(y + \Delta y/2) \;//\; \Delta x \;\&\; \Delta x = 0\} = y$ since $\Delta y = f[x + \Delta x] - f[x] = 0$ when $\Delta x = 0$. Thus, in deriving $F'[x]$ it seems that we divide equation (*) on both sides by Δx but the true point is that we use the fact that the differential quotient is a (dynamic) quotient. In this case the seeming division on both sides of equation (*) actually does not give a problem since it is a "//" again, yet the right hand side might give a problem, conceptually, since a "surface" with a vanishing width is no surface at all. The point however was to show that the derivative of a surface function F generates the function $f = F'$. The choice of $\Delta x = 0$ rather deals with finding the derivative and not with finding the surface.

Hence, the proper introduction to calculus might be to start with a function that describes a surface and then find the derivative. This establishes that the reverse operation on the derivative gives a function for the surface.

This is not to say that it wouldn't be enlightening to support an analysis like this with graphics of approximating a surface, especially to explain where $\int f[x]\, dx$ comes from. Historically, surfaces and volumes also seem more important conceptual drivers than gradients. Nevertheless, the discussion above suggests that the major stumbling block still is the derivative and the seeming division by zero, which stumbling block we also find in above explanation of integration. This has now been dealt with in a satisfactory manner.

Naturally, to be completely satisfactory, we should note that (*) actually is a bad approximation. I have hesitated to start with it but (*) is close to current practice of exposition and thus (*) provides a bridge for understanding. But properly speaking, (*) is an approximation for which we have $\approx$ instead of =. Proper is:

$$\Delta F = F[x + \Delta x] - F[x] = \Delta x\,(y + \Delta y/2) + \Delta x\; error[\Delta x] \qquad (**)$$

which is an implicit definition of the error function $error[\Delta x]$. Again we find $y = f[x] = F'[x]$, using that $error[0] = 0$.

Expression (**) also allows us to understand the derivative as a result of the explicit formulation of the error function:

$$error[\Delta x] = \Delta F \;//\; \Delta x - (y + \Delta y/2) \qquad (***)$$

The deeper reason (or "trick") for this construction is that (**) does not use an $\epsilon[\Delta x]$ which would cause the question what the outcome of $\epsilon[\Delta x]$ // Δx would be. Instead, the explicit definition (***) or implicit version (**) gives exactly what we need for both a good expression of the error and subsequently the derivative at $\Delta x = 0$.

Of course, once we know this, it is OK to use (*) for $\Delta x = 0$ as well. However, though this is standard practice, and conceivably OK for introductory courses in calculus, we could wonder what actually would be the best approach. It may well be that the complete discussion of the true error function via (**) and (***) brings much more understanding where the notion of the "derivative" comes from. Possibly some randomized controlled trials in education would bring more light in this question.

Aspect 5: Refoundation

All these are not just didactic observations but amount to an essential refoundation of calculus. The approach allows solutions where situations up to now are undefined. E.g. the derivative of $| x |$ traditionally is undefined at $x = 0$ but would algebraically become sgn[x]. Namely, for $x \neq 0$, we can work through the various combinations and find the normal result, sgn[x], while for $x = 0$ the differential quotient collapses to $| \Delta x |$ // Δx = sgn[Δx] as well, which becomes 0 when $\Delta x = 0$. One can imagine that there will be more results like this. There will be a tendency amongst mathematicians to think that calculus has been well-developed and no longer is a subject for research so that the upshot of this paper in their eyes will only be didactic. This thus would be a misconception.

Conclusion

History is a big subject and we should be careful about drawing big historical lines. But the following seems an acceptable summary of the situation where we currently find us after the introduction of the 0.

Historically, the introduction of the 0 gave so many problems that when those were getting solved, those solutions, such as that one cannot divide by zero, were codified in stone, and pupils in the schools of Europe would meet with bad grades, severe punishment and infamy if they would sin against those sacrosanct rules. Tragically, a bit later on the historical timeline, division by zero appeared important for the differential quotient. Rather than slightly modifying our concept of division, Leibniz, Newton and Weierstrasz decided to work around this, creating the concepts of infinitesimals or the limit. In this way they actually complicated the issue and created paradoxes of their own. Logical clarity and soundness can be restored by distinguishing between the act of division and the result of division.

"Most" derivatives can be found without the Weierstrasz technical purity and "many" courses teach calculus without developing that purity. The distinction between static and dynamic division suggests that the Weierstrasz purity may be overly pedantic for the main body of calculus. The exact definition of the limit is of great value but not necessarily for calculus. The Weierstrasz $\epsilon > 0$ and $\delta > 0$ and the derivative's shorthand $lim(\Delta x \to 0)\ \Delta f / \Delta x$ are paradoxical since those exclude the zero values that are precisely the values of interest at the point where the limit is taken. Instead, using Δf // Δx and then setting $\Delta x = 0$ is not paradoxical at all. "Much" of calculus might well do without the limit idea.

Literature

Colignatus is the name of Thomas Cool in science.

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